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A181875/A181876. Minimal Polynomials of $\cos\left(\frac{2\pi}{n}\right)$

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The minimal polynomial of an algebraic number α of degree d_α is the monic, minimal degree rational polynomial which has as root, or as one of its roots, α. This minimal degree d_α is 1 iff α is rational, and the minimal polynomial in this case is $p(x) = x - \alpha$. For the notion 'minimal polynomial of an algebraic number' see, *e.g.*, [6], p. 28.

For the algebraic number $\cos\left(\frac{2\pi}{n}\right)$, for $n \in \mathbb{N}$, the degree (called here $d(n)$) is $d(1) = 1$, $d(2) = 1$, and $d(n) = \frac{\varphi(n)}{2}$, with *Euler*'s totient function $\varphi(n) =$ A000010(n). See [3], and [6], Theorem 3.9, p. 37. In [7] one finds the degree sequence as $d(n) =$ A023022(n) , $n \geq 2$, with $d(1) = 1$. These minimal polynomials of *cos* $\left(\frac{2\pi}{n}\right)$ have been discussed in [9] where they have been called $\Psi_n(x)$. We will call them $\Psi(n,x)$, and give a list of the first 30 polynomials in *Table 1*, as well as the numerator and denominator arrays of the coefficients A181875(n,m) and A181876(n,m) in *Table 2* and *Table 3*, respectively. The rational coefficients for the monic polynomials $\Psi(n,x)$ will be given in *Table 4*. *Table 5* shows the head of the integer coefficient array of the non-monic $\psi(n,x) := 2^{d(n)}\,\Psi(n,x)$ polynomials. This is A181877(n,m).

In [9] one finds a recurrence relation for the minimal polynomials $\Psi(n,x)$ based on *Chebyshev*'s $T-$polynomials. We give now a generic example for the application of this recurrence. The example $\Psi(9,x)$ has been treated in the mentioned reference. Assume that one wishes to compute $\Psi(n,28)$. First consider the list of divisors of 28, *viz* $[1,2,4,7,14,28]$. According to reference [9], eq. (3), one needs, what we call $Tnf(n,x)$, which is the factorized form of $(T(\frac{n}{2}+1,x) - T(\frac{n}{2}-1,x))/2^{\frac{n}{2}}$ if n is even, and of $(T(\frac{n+1}{2},x) - T(\frac{n-1}{2},x))/2^{\frac{n-1}{2}}$ if n is odd. The formula which leads to the recurrence is $Tnf(n,x) = \prod_{d|n} \Psi(n,d)$, with the divisors d of n. Now we have

$$\Psi(28,x) \;=\; \frac{Tnf(28,x)\,Tnf(2,x)}{Tnf(14,x)\,Tnf(4,x)}\,. \tag{1}$$

The numerators and denominators follow from the divisors of 28 and the $Tnf(n,x)$ formula in terms of the Ψ-products. $Tnf(28,x)$ has besides the wanted $\Psi(n,28)$ also $\Psi(1,x)$, $\Psi(2,x)$, $\Psi(4,x)$, $\Psi(7,x)$, and $\Psi(14,x)$. Therefore one divides $Tnf(14,x)$, which, in excess of $\Psi(14,x)$ has also the factors $\Psi(1,x)$, $\Psi(2,x)$, $\Psi(7,x)$. In order to divide the remaining $\Psi(4,x)$ one divides $Tnf(4,x)$, which, however, also has as factors $\Psi(1,x)$ and $\Psi(2,x)$, which is $Tnf(2,x)$, and this then appears in the numerator. This kind of compensation procedure works in general, and the results given in *Table 1* have been found by a Maple [5] program, based on the Tnf-formula from reference [9]. See also the W. Lang link under A007955 which deals with the (unique) representation of any natural number in terms of products of divisors which is used in this program. The answer for the example is thus

$$\psi(28,x) \;=\; 64\,x^6 \,-\, 112\,x^4 \,+\, 56\,x^2 \,-\, 7\;, \tag{2}$$

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or

$$\Psi(28,x) = x^6 - \frac{7}{4}x^4 + \frac{7}{8}x^2 - \frac{7}{64}\,. \tag{3}$$

Due to the *Lemma* found on p. 473 of [9] the linear factors of the minimal polynomials are

$$\Psi(n,x) = \prod_{\substack{k=0\\ gcd(k,n)=1}}^{\lfloor(\rfloor \frac{n}{2})} \left(x - \cos\left(\frac{2\,\pi\,k}{n}\right)\right)\,, \tag{4}$$

where $gcd(k,n)$ is the greatest common divisor of k and n. Remember that $gcd(0,n) = n$, for $n \geq 1$, hence $k = 0$ is not allowed for $n \geq 2$. Even though this looks like a non-rational polynomial in general, it is in fact rational due to the *Lemma*. For example, the two zeros of $\Psi(5,x)$ are $\cos(2\,\pi/5) = \frac{\phi - 1}{2}$ and $\cos\left(\frac{4\,\pi}{5}\right) = -\frac{\phi}{2}$, with $\phi := \frac{1}{2}(1 + \sqrt{5})$ (the golden section). Therefore, $\Psi(5,x) = (x - \frac{\phi - 1}{2})\,(x + \frac{\phi}{2}) = x^2 + \frac{1}{2}x - \frac{1}{4}$, due to the property $(\phi - 1)\,\phi = 1$, rendering a rational polynomial $\Psi(5,x)$.

The minimal polynomials $\Psi(n,x)$, for n=1,..,30, have been listed as a comment on A023022 (the degree sequence) by *Artur Jasinski*. They are given here in *Table 1* in falling powers of x.

Note added Feb 23 and 25 2011

Gary Detlefs noticed, in an e-mail to the author, for some instances that $\Psi(n,\cos(x))$ can be written as a sum over $cos(k\,x)$. His observation generalizes to the following formulae for $\Psi(n,x)$ for prime numbers $n = p$. We use the $\Psi(n,x)$ formula given in the W. Lang link under A007955 which resulted from the recurrence relation given in [9]. There one finds the definition of $t(n,x)$, and we will use *Chebyshev*'s $T-$ and $U-$polynomials.

$$2\,\Psi(2,x) \;=\; 2\,\frac{t(2,x)}{t(1,x)} = \frac{T(2,x) - T(0,x)}{T(1,x) - T(0,x)} = 2\sum_{l=0}^{1} T(l,x) = U(1,x) + 2. \tag{5}$$

$$2^k\,\Psi(p,x) \;=\; 2^k\,\frac{t(p,x)}{t(1,x)} = \frac{T(k+1,x) - T(k,x)}{T(1,x) - T(0,x)} = 2\sum_{l=0}^{k} T(l,x) - 1 = \tag{6}$$

$$U(k,x) + U(k-1,x),\ \text{with } p = 2\,k + 1,\ \text{ prime.} \tag{7}$$

Proof:

First consider the sum over *Chebyshev*'s T-polynomials. This reduces (after an index shift) to a telescopic sum when the 'trace' formula $T(n,x) = (U(n,x) - U(n-2,x))/2$ is used. Then multiply the denominator $T(1,x) - 1$ with the result of the sum, *i.e.* $U(k,x) + U(k-1,x)$, and use the following well known formula $2\,T(n,x)\,U(m,x) = U(m+n,x) + U(m-n,x)$, for $m \geq n$ (the formula for $m < n$ is $2\,T(n,x)\,U(m,x) = U(m+n,x) + U(n-m-2,x)$, with $U(-1,x) := 0$). This then yields $T(k+1,x) - T(k,x)$ if one applies the 'trace' formula for the $T-$polynomials. The case $p = 2$ works the same way, but the -1 in the sum is not present. $\square$

This generalizes to the following

Proposition:

For powers of prime numbers p from A000040 the minimal polynomials $\Psi(p^m,x)$ can be written in the following forms involving *Chebyshev*'s U (or S) and T-polynomials A049310 and A053120, respectively.

a) If $p = 2$ then

$$2^{2^{m-2}}\,\Psi(2^m,x) \;=\; \frac{U(2^{m-1}-1,x)}{U(2^{m-2}-1,x)} \;=\; 2\,T(2^{m-2},x)\;,\;\; m\,\in\,\{2,3,...\}\,. \tag{8}$$

b) For odd primes $p \,=\, 2\,k+1$ one finds for $m\,\in\,\mathbb{N}$

$$2^{p^{m-1}\,(p-1)/2}\,\Psi(p^m,x) \;=\; \frac{T(\frac{p^m+1}{2},x)\;-\;T(\frac{p^m+1}{2}-1,x)}{T(\frac{p^{m-1}+1}{2},x)\;-\;T(\frac{p^{m-1}+1}{2}-1,x)} \;= \tag{9}$$

$$2\sum_{j=1}^{k} T(p^{m-1}\,\frac{p-(2j-1)}{2},x)\;+\;1 \;=\; 2\sum_{l=1}^{k} T(p^{m-1}\,l,x)\;+\;1\;. \tag{10}$$

Proof: One starts with the general formula for $\Psi(n,x)$ given in the W. Lang link under A007955 in eq. (1). The powers of 2 there are collected and written in front of $\Psi(n,x)$. The divisor product representation for powers p^m is $dpr(p^m) \;=\; \dfrac{a(p^m)}{a(p^{m-1})}$ with the divisor products $a(k)$. This determines the formula for $\Psi(p^m,x)$ in terms of a quotient of $T-$polynomials which is rewritten in this *proposition*.

a) One has to use $m\,\geq\,2$ for the following. Here $2^{2^{m-2}}\,\Psi(2^m,x) \;=\; \dfrac{T(2^{m-1}+1,x)\;-\;T(2^{m-1}-1,x)}{T(2^{m-2}+1,x)\;-\;T(2^{m-2}-1,x)}$ is rewritten with the help of the known formula $T(n+1,x)\;-\;T(n-1,x) \;=\; 2\,(x^2-1)\,U(n-1,x)\,U(0,x) \;=\; 2\,(x^2-1)\,U(n-1,x)$ (see *e.g.*, [4], p.261, 1st line). This produces (certainly for $x^2\,\neq\,1$, but it is also true for these values) $\dfrac{U(2^{m-1}-1,x)}{U(2^{m-2}-1,x)}$. Then one uses the known identity $2\,T(n,x)\,U(n-1,x) \;=\; U(2\,n-1,x)$ (see *e.g.*, [4], p.260, last line) which will produce the assertion.

b) This is more involved and uses the identity $2\,T(n,x)\,T(m,x) \;=\; T(n+m,x)\;+\;T(n-m,x)$ if $n\,\geq\,m$ (see *e.g.*, [4], p.260, 5.7.3. 1st formula). The general formula is given in the first equation of the *proposition*. One shows that the numerator can be written as the sum given as second eq. multiplied by the denominator. This will result in a telescopic summation with the first term just the two numerator terms and the last term the negative of the two denominator terms. Hence, when the $+1$ term after the summation is used, one is left with just the numerator terms. We give an example for this cancellation mechanism before going into the proof:

$$p=5,(k=2),m=4:\;\; 2^{250}\,\Psi(5^4) \;=\; \frac{T(313,x)\;-\;T(312,x)}{T(63,x)\;-\;T(62,x)}\;. \tag{11}$$

$$(2\,T(250,x)\;+\;2\,T(125,x)\;+\;1)\;(T(63,x)\;-\;T(62,x))\;= \tag{12}$$

$$\mathbf{T(313,x)}\;-\;\mathbf{T(312,x)}\;+\;T(187,x)\;-\;T(188,x)\;+ \tag{13}$$

$$T(125+63,x)\;-\;T(187,x)\;+\;T(125-63,x)\;-\;T(63,x) \tag{14}$$

$$+T(63,x)\;-\;T(62,x)\;. \tag{15}$$

This kind of telescoping works also in the general case. The first term of the sum is $2\,T(a_{m,1},x)$ with $a_{m,1}\;:=\;p^{m-1}\,\frac{p-1}{2}$ which when multiplied with the denominator $T(b_m,x)\;-\;T(b_m-1,x)$, with $b_m\;:=\;\frac{p^{m-1}+1}{2}$, becomes the numerator because $a_{m,1}\;+\;b_m\;=\;\frac{p^m+1}{2}$, and a remainder $T(a_{m,1}\;-\;b_m,x)\;-\;T(a_{m,1}-b_m+1,x)$. The second term of the sum, after multiplication, produces an argument $a_{m,2}+b_m$, with $a_{m,2}\;:=\;p^{m-1}\,\frac{p-3}{2}$ which is in fact $a_{m,1}\;-\;b_m\;+\;1$. Therefore, the second term of the sum cancels the remainder from the multiplication of the first term of the sum, and produces a new remainder , then canceled by the first two terms after multiplication of the third term from the sum, *etc.*. The last term of the sum then has, after multiplication, a remainder which is $T(b_m\;-\;1,x)\;-\;T(b_m,x)$, due to $p^{m-1}\cdot 1\;-\;b_m\;=\;b_m\;-\;1$. This remainder is canceled by the $+1$ term when multiplied with $T(b_m,x)\;-\;T(b_m\;-\;1,x)$.

□

On Feb 25 2011 I found a paper by *D. Surowski* and *P. McCombs* [8] on the web where (contrary to the title) the minimal polynomial of $\mathbf{2}\,\cos\left(\frac{2\,\pi}{p}\right)$ for odd primes p has been computed in Theorem 3.1, where it is called $\Theta_p(x)$. There is a misprint: σ_{2k-1}, not σ_{2k+1}. The relation to the notation here is (see the *proposition* for odd p and $m = 1$) $\Theta_p(2\,x) \;=\; 2^{(p-1)/2}\,\Psi(p,x)$.

On Feb 26 2011 I found the paper by Chan-Lye Lee and K. B. Wong [2] with factorizations of *Chebyshev*'s $U(2n-1,x)$ and $U(2n,x)$ polynomials, and references to the minimal polynomial papers [8] and [1].
The paper by *S. Beslin* and *V. de Angelis* [1] gives correct formulae for the (integer) minimal polynomials of $\sin\left(\frac{2\,\pi}{p}\right)$ and $\cos\left(\frac{2\,\pi}{p}\right)$ for odd primes.

References

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Table 1: **Minimal polynomials of** $\cos\left(\frac{2\pi}{n}\right)$ **for** $n = 1, 2, ..., 30$.

n	$\Psi(n, x)$
1	$x-1$
2	$x+1$
3	$x+1/2$
4	x
5	$x^2+(1/2)\,x-1/4$
6	$x-1/2$
7	$x^3+(1/2)\,x^2-(1/2)\,x-1/8$
8	$x^2-1/2$
9	$x^3-(3/4)\,x+1/8$
10	$x^2-(1/2)\,x-1/4$
11	$x^5+(1/2)\,x^4-x^3-(3/8)\,x^2+(3/16)\,x+1/32$
12	$x^2-3/4$
13	$x^6+(1/2)\,x^5-(5/4)\,x^4-(1/2)\,x^3+(3/8)\,x^2+(3/32)\,x-1/64$
14	$x^3-(1/2)\,x^2-(1/2)\,x+1/8$
15	$x^4-(1/2)\,x^3-x^2+(1/2)\,x+1/16$
16	$x^4-x^2+1/8$
17	$x^8+(1/2)\,x^7-(7/4)\,x^6-(3/4)\,x^5+(15/16)\,x^4+(5/16)\,x^3-(5/32)\,x^2-(1/32)\,x+1/256$
18	$x^3-(3/4)\,x-1/8$
19	$x^9+(1/2)\,x^8-2\,x^7-(7/8)\,x^6+(21/16)\,x^5+(15/32)\,x^4-(5/16)\,x^3-(5/64)\,x^2+(5/256)\,x+1/512$
20	$x^4-(5/4)\,x^2+5/16$
21	$x^6-(1/2)\,x^5-(3/2)\,x^4+(3/4)\,x^3+(1/2)\,x^2-(1/4)\,x+1/64$
22	$x^5-(1/2)\,x^4-x^3+(3/8)\,x^2+(3/16)\,x-1/32$
23	$x^{11}+(1/2)\,x^{10}-(5/2)\,x^9-(9/8)\,x^8+(9/4)\,x^7+(7/8)\,x^6-(7/8)\,x^5-(35/128)\,x^4+(35/256)\,x^3+$ $(15/512)\,x^2-(3/512)\,x-1/2048$
24	$x^4-x^2+1/16$
25	$x^{10}-(5/2)\,x^8+(35/16)\,x^6+(1/32)\,x^5-(25/32)\,x^4-(5/128)\,x^3+(25/256)\,x^2+(5/512)\,x-1/1024$
26	$x^6-(1/2)\,x^5-(5/4)\,x^4+(1/2)\,x^3+(3/8)\,x^2-(3/32)\,x-1/64$
27	$x^9-(9/4)\,x^7+(27/16)\,x^5-(15/32)\,x^3+(9/256)\,x+1/512$
28	$x^6-(7/4)\,x^4+(7/8)\,x^2-7/64$
29	$x^{14}+(1/2)\,x^{13}-(13/4)\,x^{12}-(3/2)\,x^{11}+(33/8)\,x^{10}+(55/32)\,x^9-(165/64)\,x^8-(15/16)\,x^7+$ $(105/128)\,x^6+(63/256)\,x^5-(63/512)\,x^4-(7/256)\,x^3+(7/1024)\,x^2+(7/8192)\,x-1/16384$
30	$x^4+(1/2)\,x^3-x^2-(1/2)\,x+1/16$
$\vdots$	

Table 2: A181875$(\mathbf{n}, \mathbf{m})$ array for numerators of coefficients of minimal polynomials of $\cos\left(\frac{2\pi}{n}\right)$

n/m	0	1	2	3	4	5	...
1	-1	1					
2	1	1					
3	1	1					
4	0	1					
5	-1	1	1				
6	-1	1					
7	-1	-1	1	1			
8	-1	0	1				
9	1	-3	0	1			
10	-1	-1	1				
11	1	3	-3	-1	1	1	
⋮							

Table 3: A181876(n, m) array for denominators of coefficients of minimal polynomials of $\cos\left(\frac{2\pi}{n}\right)$

n/m	0	1	2	3	4	5	...
1	1	1					
2	1	1					
3	2	1					
4	1	1					
5	4	2	1				
6	2	1					
7	8	2	2	1			
8	2	1	1				
9	8	4	1	1			
10	4	2	1				
11	32	16	8	1	2	1	
⋮							

Table 4: **A181875**(n, m)/**A181876**(n, m) array for coefficients of minimal polynomials of $\cos\left(\frac{2\pi}{n}\right)$

n/m	0	1	2	3	4	5	...
1	-1	1					
2	1	1					
3	1/2	1					
4	0	1					
5	-1/4	1/2	1				
6	-1/2	1					
7	-1/8	-1/2	1/2	1			
8	-1/2	0	1				
9	1/8	-3/4	0	1			
10	-1/4	-1/2	1				
11	1/32	3/16	-3/8	-1	1/2	1	
⋮							

Table 5: A181877$(\mathbf{n},\mathbf{m})$ **array for integer coefficients of** $\boldsymbol{\psi}(\mathbf{n},\mathbf{x}) := \mathbf{2}^{\mathbf{d}(\mathbf{n})}\, \boldsymbol{\Psi}(\mathbf{n},\mathbf{x})$.

n/m	0	1	2	3	4	5	...
1	-2	2					
2	2	2					
3	1	2					
4	0	2					
5	-1	2	4				
6	-1	2					
7	-1	-4	4	8			
8	-2	0	4				
9	1	-6	0	8			
10	-1	-2	4				
11	1	6	-12	-32	16	32	
⋮							