COUNTING BINARY MATRICES WITH EVERY 3×3 BLOCK HAVING EXACTLY FOUR ONES

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1. Introduction

In this article, we derive a recurrence for the number of $(n+2) \times (k+2)$ binary matrices with $n \geq 1$ and $k \geq 1$ such that every 3×3 block has exactly four ones. This is Sequence A181262 in the On-Line Encyclopedia of Integer Sequences. For brevity, let us say that a matrix is *good* if it satisfies the conditions described above.

A binary matrix is an integer matrix in which each entry is either zero or one. A 3×3 block of a matrix $A = (a_{i,j})$ is a submatrix A' of the form

$$A' = \begin{pmatrix} a_{i,j} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{pmatrix}.$$

We will show that, for each fixed value of k, the number of solutions $\boldsymbol{s}$ satisfies the following 24th-order recurrence relation:

Theorem 1. *Let* $A = \{2, 3, 4, 6, 9, 12, 18\}$*. Then*

$$(T - I) \circ (T - 2I) \circ (T - 3I) \circ \prod_{a \in A} (T^3 - aI)(\boldsymbol{s}) = \mathbf{0}.$$

The notation will be explained in the next section.

2. Review of sequences and linear operators

Let $\mathbb{R}^\omega$ denote the set of all infinite sequences of real numbers. $\mathbb{R}^\omega$ is a real vector space, with addition and scalar multiplication defined elementwise.

$$(s_1, s_2, \ldots) + (t_1, t_2, \ldots) = (s_1 + t_1, s_2 + t_2, \ldots)$$
$$c \cdot (s_1, s_2, \ldots) = (cs_1, cs_2, \ldots)$$

The zero sequence

$$\mathbf{0} = (0, 0, \ldots)$$

is the additive identity for $\mathbb{R}^\omega$.

A linear operator on $\mathbb{R}^\omega$ is a function $L : \mathbb{R}^\omega \to \mathbb{R}^\omega$ such that $L(\boldsymbol{s} + \boldsymbol{t}) = L(\boldsymbol{s}) + L(\boldsymbol{t})$ and $L(c \cdot \boldsymbol{s}) = c \cdot L(\boldsymbol{s})$ for all $\boldsymbol{s}, \boldsymbol{t} \in \mathbb{R}^\omega$ and $c \in \mathbb{R}$. Important examples of linear operators on $\mathbb{R}^\omega$ include:

(1) The identity operator I, defined by $I(\boldsymbol{s}) = \boldsymbol{s}$ for all $\boldsymbol{s} \in \mathbb{R}^\omega$, and

(2) The shift operator T, defined by $T(\boldsymbol{s})(n) = \boldsymbol{s}(n+1)$ for all $\boldsymbol{s} \in \mathbb{R}^\omega$ and $n \in \mathbb{N}$.

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The set of all linear operators on $\mathbb{R}^\omega$ is closed under addition, scalar multiplication, and composition. Nonnegative integer powers of a linear operator are defined via iterated composition:

$$L^n = \begin{cases} I & \text{if n} = 0, \\ L \circ L^{n-1} & \text{if } n \geq 1. \end{cases}$$

Composition of linear operators is not commutative in general; but polynomial functions of the same operator commute with one another. That is, if $P = \sum_{i=0}^m a_i L^i$ and $Q = \sum_{j=0}^n b_j L^j$, then $P \circ Q = Q \circ P$. This follows from the distributive law and the fact that $L^i \circ L^j = L^j \circ L^i = L^{i+j}$.

The composition of several linear operators $L_1 \circ \cdots \circ L_k$ is denoted by $\prod_{i=1}^k L_i$.

Theorem 2. *Let $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_r \in \mathbb{R}^\omega$, and let $L_1, \ldots, L_r$ be pairwise commuting linear operators on $\mathbb{R}^\omega$ such that $L_i(\boldsymbol{s}_i) = \mathbf{0}$ for $i = 1, \ldots, r$. Then*

$$\left(\prod_{i=1}^r L_i\right)\left(\sum_{j=1}^r \boldsymbol{s}_j\right) = \mathbf{0}.$$

Proof. Since the L_i commute with one another,

$$\left(\prod_{i=1}^r L_i\right)(\boldsymbol{s}_j) = \left(\left(\prod_{i\neq j} L_i\right) \circ L_j\right)(\boldsymbol{s}_j) = \left(\prod_{i\neq j} L_i\right)(\mathbf{0}) = \mathbf{0}.$$

Therefore,

$$\left(\prod_{i=1}^r L_i\right)\left(\sum_{j=1}^r \boldsymbol{s}_j\right) = \sum_{j=1}^r \left(\prod_{i=1}^r L_i\right)(\boldsymbol{s}_j) = \sum_{j=1}^r \mathbf{0} = \mathbf{0}.$$

□

3. An equivalence relation

Define an equivalence relation $\sim$ on binary sequences of length $k+2$ as follows:

$$(s_1, \ldots, s_{k+2}) \sim (t_1, \ldots, t_{k+2}) \iff \sum_{i=j}^{j+2} s_i = \sum_{i=j}^{j+2} t_i \quad (\forall j, 1 \leq j \leq k).$$

Theorem 3. *Every equivalence class of $\sim$ has at most three elements.*

Proof. Consider the following system of equations in $a_1, \ldots, a_{k+2}$, where $c_1, \ldots, c_{k+2}$ are given:

$$\begin{cases} a_1 & = c_1 \\ a_2 & = c_2 \\ a_1 + a_2 + a_3 & = c_3 \\ & \vdots \\ a_k + a_{k+1} + a_{k+2} & = c_{k+2}. \end{cases}$$

This system has a unique real solution, since it is lower triangular with nonzero coefficients along the diagonal.

Delete the first two equations. Then we can solve for the remaining variables $a_3, \ldots, a_{k+2}$ in terms of a_1 and a_2. Since there are four possible combinations of binary values for a_1 and a_2, the system has at most four binary solutions.

Consider the equation $a_1 + a_2 + a_3 = c_3$. If $(a_1, a_2) = (0, 0)$ then c_3 must be 0 or 1, while if $(a_1, a_2) = (1, 1)$ then c_3 must be 2 or 3. At most one of these statements can be true. Therefore, the system has at most three binary solutions, so each equivalence class has at most three elements. $\square$

Note: We can easily find equivalence classes having exactly 1, 2, or 3 elements. The sequence $(0, 0, 0)$ is equivalent only to itself. The sequence $(0, 0, 1, 1)$ is equivalent only to itself and $(0, 1, 0, 1)$. Finally, $(0, 0, 1)$ is equivalent to $(0, 1, 0)$ and $(1, 0, 0)$.

For any matrix A, let $A^{(i)}$ denote the i-th row of A. We extend $\sim$ to an equivalence relation $\approx$ on binary matrices having at least three rows via $A \approx B \iff A^{(i)} \sim B^{(i)}$ for $i = 1, 2, 3$. Note that equivalent matrices may have different numbers of rows. There are only finitely many equivalence classes for a given number of columns, since equivalence is determined by the first three rows.

Theorem 4. *Let A and B be binary matrices having the same number of columns. Suppose that A is good and $A \approx B$. Then B is good if and only if $B^{(i)} \sim B^{(i+3)}$ for all i, $1 \le i \le n-1$ where $n+2$ is the number of rows of B.*

Proof. Assume that $B = (b_{i,j})$ is good. For any i, j with $1 \le i \le n-1$ and $1 \le j \le k$, we have

$$\sum_{u=i}^{i+2} \sum_{v=j}^{j+2} b_{u,v} = 4 \quad \text{and} \quad \sum_{u=i+1}^{i+3} \sum_{v=j}^{j+2} b_{u,v} = 4.$$

Subtracting these equations yields

$$\sum_{v=j}^{j+2} b_{i,v} = \sum_{v=j}^{j+2} b_{i+3,v}$$

for all v. Therefore, $B^{(i)} \sim B^{(i+3)}$.

Conversely, suppose that $B^{(i)} \sim B^{(i+3)}$ for all i, $1 \le i \le n-1$. It is required to prove that

$$\sum_{u=i}^{i+2} \sum_{v=j}^{j+2} b_{u,v} = 4 \tag{1}$$

for all i, j with $1 \le i \le n-1$ and $1 \le j \le k$.

Equation 1 holds when $i = 1$ because the first three rows of B are equivalent to the first three rows of A, so

$$\sum_{u=1}^{3} \sum_{v=i}^{i+2} b_{u,v} = \sum_{u=1}^{3} \sum_{v=i}^{i+2} a_{u,v} = 4.$$

Assume that Equation 1 holds for some $i = i_0$ with $1 \le i_0 \le n-2$. This implies that

$$\sum_{u=i_0}^{i_0+2} \sum_{v=j}^{j+2} b_{u,v} = 4 \tag{2}$$

for all j with $1 \le j \le k$.

Since $B^{(i_0)} \sim B^{(i_0+3)}$,

$$\sum_{v=j}^{j+2} (b_{i_0+3,v} - b_{i_0,v}) = 0. \tag{3}$$

Adding Equations 2 and 3 yields

$$\sum_{u=i_0+1}^{i_0+3} \sum_{v=j}^{j+2} b_{u,v} = 4.$$

Therefore, Equation 1 holds for $i = i_0 + 1$, so it holds for all i by induction. □

Theorem 5. *Let A be a good $3 \times (k+2)$ matrix, and let $\boldsymbol{s}(n)$ denote the number of good $(n+2) \times (k+2)$ matrices that are equivalent to A. Let c_1, c_2, and c_3 denote the cardinalities of the equivalence classes of the rows of A. Then $\boldsymbol{s}(n+3) = c_1c_2c_3\boldsymbol{s}(n)$ for all $n \geq 1$. Moreover, if $c_1 = c_2 = c_3$ then $\boldsymbol{s}(n+1) = c_1\boldsymbol{s}(n)$.*

Proof. Let B a good $(n+2)\times(k+2)$ binary matrix that is equivalent to A By the previous theorem, we have

$$\begin{aligned} A^{(1)} &\sim B^{(1)} \sim B^{(4)} \sim B^{(7)} \sim \cdots \\ A^{(2)} &\sim B^{(2)} \sim B^{(5)} \sim B^{(8)} \sim \cdots \\ A^{(3)} &\sim B^{(3)} \sim B^{(6)} \sim B^{(9)} \sim \cdots \end{aligned}$$

There are c_1 ways to choose each of the rows $B^{(1)}, B^{(4)}, B^{(7)}$, etc. Similarly, there are c_2 ways to choose each of the rows $B^{(2)}, B^{(5)}, B^{(8)}$, etc., and c_3 ways to choose each of the rows $B^{(3)}, B^{(6)}, B^{(9)}$, etc. Conversely, any combination of these choices yields a good matrix that is equivalent to A.

Therefore, $\boldsymbol{s}(n) = \underbrace{c_1c_2c_3c_1c_2c_3\cdots}_{n+2 \text{ factors}}$ which implies that $\boldsymbol{s}(n+3) = c_1c_2c_3\boldsymbol{s}(n)$.

In the special case where $c_1 = c_2 = c_3$, we find that $\boldsymbol{s}(n) = c_1^{n+2}$, which implies that $\boldsymbol{s}(n+1) = c_1\boldsymbol{s}(n)$.

□

4. Proof of Theorem 1

Let $k \geq 1$ be a fixed integer, and let $A_1, \ldots, A_r$ be a complete set of representatives for the equivalence relation $\approx$ on all $3 \times (k+2)$ binary matrices.

For each $n \geq 1$, let $\boldsymbol{s}(n)$ denote the number of good $(n+2)\times(k+2)$ matrices, and let $\boldsymbol{s}_i(n)$ denote the number of good $(n+2) \times (k+2)$ matrices that are equivalent to A_i. Note that $\boldsymbol{s} = \sum_{i=1}^{r} \boldsymbol{s}_i$, since each good matrix is equivalent to exactly one A_i.

For each i with $1 \leq i \leq r$, we define a linear operator L_i that satisfies $L_i(\boldsymbol{s}_i) = \mathbf{0}$. Let c_1, c_2, and c_3 denote the cardinalities of the equivalence classes of the rows of A_i, and define

$$L_i = \begin{cases} T - c_1 I & \text{if } c_1 = c_2 = c_3, \\ T^3 - c_1c_2c_3 I & \text{otherwise.} \end{cases}$$

Then $L_i(\boldsymbol{s}_i) = \mathbf{0}$ by Theorem 5.

Since $c_1, c_2, c_3 \in \{1, 2, 3\}$ by Theorem 3, there are only 10 possible values for L_i, namely $T - I$, $T - 2I$, $T - 3I$, and $T^3 - aI$ for $a \in \{2, 3, 4, 6, 9, 12, 18\}$. All of these operators commute with one another, since they are polynomials in T.

Let $A = \{2, 3, 4, 6, 9, 12, 18\}$ and let

$$L = (T - I) \circ (T - 2I) \circ (T - 3I) \circ \prod_{a \in A} (T^3 - aI).$$

Then $L(\boldsymbol{s}) = \mathbf{0}$ by Theorem 2, which completes the proof.