Proofs of some formulae on the number of k-reverses of n using some formulae by A. Howroyd

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In this note, we prove two formulae about $R(n,k)$, the number of k-reverses of n (where $1 \le k \le n$). These numbers appear in sequence A180171 in the OEIS. As noted in the comments for the sequence, the "reverse of a k-composition is the k-composition obtained by writing its parts in reverse." In addition, a "k-reverse of n is a k-composition of n which is cyclically equivalent to its reverse."

The equivalence classes created by cyclically equivalent k-reverses of n are called "Sommerville symmetrical cyclic compositions" because they were studied by Sommerville (1909, pp. 301–304). The number $T(n,k)$ of Sommerville symmetrical cyclic compositions of n with k parts (or equivalently, the number of equivalence classes of cyclically equivalent k-compositions of n) appear in sequence A119963 (for $1 \le k \le n$ by ignoring the numbers $T(n,0)$).

Sommerville (1909) proved that, for $0 \le k \le n$ (with the exception of the case $T(0,0)$),

$$T(2n,2k) = T(2n+1,2k) = T(2n+1,2k+1) = T(2n+2,2k+1) = \binom{n}{k}.$$

More than a century later, these formulae were re-discovered (in a slightly different context) by McSorley and Shoen (2013).

Let A is a set of positive integers and, for $1 \le k \le n$, let $T_A(n,k)$ be the total number of Sommerville symmetrical cyclic compositions of n with length k and parts only in A (that is, the number of equivalence classes of cyclically equivalent k-reverses of n with parts only in A). Hadjicostas and Zhang (2017) proved that the g.f. of $T(n,k)$ is

$$\sum_{n,k\ge 1} T_A(n,k)\, x^n y^k = \frac{(1+yf_A(x))^2}{2(1-y^2 f_A(x^2))} - \frac{1}{2}, \tag{1}$$

where $f_A(x) = \sum_{m\in A} x^m$. For sequence A119963, A = all positive integers = $\mathbb{Z}_{>0}$, in which case, $T_A(n,k) = T(n,k)$ and equation (1) becomes

$$\sum_{n,k\ge 1} T(n,k)\, x^n y^k = \frac{(1+xy-x^2)xy}{(1-x)(1-x^2-x^2y^2)}. \tag{2}$$

Let $\mathrm{AR}(n,k)$ be the number of aperiodic k-reverses of n. These numbers appear in sequence A180279. According to the documentation of the sequence, a "k-composition is aperiodic (primitive) if its period is k, or if it is not the concatenation of at least two smaller [equal] compositions."

In October 2017, A. Howroyd provided the following two formulae in the documentation of sequences A180171 and A180279:

$$R(n,k) = \sum_{d|\gcd(n,k)} \mathrm{AR}\left(\frac{n}{d}, \frac{k}{d}\right) \quad \text{and} \quad \mathrm{AR}(n,k) = k \sum_{d|\gcd(n,k)} \mu(d)\, T\left(\frac{n}{d}, \frac{k}{d}\right), \tag{3}$$

valid for $1 \le k \le n$. Here, $\mu(d)$ is the Möbius function at positive integer d, given by sequence A008683.

Based on the two formulae above, we shall prove the following two formulae:

$$R(n,k) = \sum_{d|\gcd(n,k)} \phi^{(-1)}(d)\, \frac{k}{d}\, T\left(\frac{n}{d}, \frac{k}{d}\right) \tag{4}$$

and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} R(n,k)\, x^n\, y^k = \sum_{s=1}^{\infty} \phi^{(-1)}(s) g(x^s, y^s), \tag{5}$$

where $\phi^{(-1)}(s)$ is the Dirichlet inverse of the Euler totient function at positive integer n, given by sequence A023900, and

$$g(x,y) = \frac{(xy+x+1)(xy-x+1)(x+1)xy}{(x^2y^2+x^2-1)^2}. \tag{6}$$

Proof. In Howroyd's equations (3), let $a = \gcd(n,k)$, $n^* = n/a$, and $k^* = k/a$. We then get:

$$R(n^*a, k^*a) = \sum_{d|a} \mathrm{AR}\left(\frac{n^*a}{d}, \frac{k^*a}{d}\right) \text{ and } \mathrm{AR}(n^*a, k^*a) = k^*a \sum_{d|a} \mu(d)\, T\left(\frac{n^*a}{d}, \frac{k^*a}{d}\right). \tag{7}$$

From the first equation in (7), we get $R(n^*a, k^*a) = \sum_{d|a} \mathrm{AR}(n^*d, k^*d)$. Using the last equation and the second equation in (7), we get

$$R(n^*a, k^*a) = \sum_{d|a} k^*d \sum_{m|d} \mu(m)\, T\left(\frac{n^*d}{m}, \frac{k^*d}{m}\right) = k^*a \sum_{d|a} \frac{d}{a} \sum_{m|d} \mu(m)\, T\left(\frac{n^*d}{m}, \frac{k^*d}{m}\right).$$

Using the associativity of Dirichlet convolutions, we get

$$R(n^*a, k^*a) = k^*a \sum_{d|a} \left(\sum_{m|d} \frac{\mu(m)}{d/m} \right) T\left(\frac{n^*a}{d}, \frac{k^*a}{d}\right) = k^*a \sum_{d|a} \frac{1}{d} \left(\sum_{m|d} m\mu(m) \right) T\left(\frac{n^*a}{d}, \frac{k^*a}{d}\right).$$

From the documentation of sequence A023900 we know that $\sum_{m|d} m\mu(m) = \phi^{(-1)}(d)$, and hence

$$R(n^*a, k^*a) = k^*a \sum_{d|a} \frac{\phi^{(-1)}(d)}{d} T\left(\frac{n^*a}{d}, \frac{k^*a}{d}\right),$$

from which we can easily prove equation (4).

To prove equation (5), we use equation (4):

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} R(n,k)\, x^n\, y^k = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \sum_{d|\,\gcd(n,k)} \phi^{(-1)}(d)\, \frac{k}{d}\, T\left(\frac{n}{d}, \frac{k}{d}\right) x^n y^k.$$

Letting $m = n/d$ and $\ell = k/d$, we then obtain

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} R(n,k)\, x^n\, y^k = \sum_{d=1}^{\infty} \phi^{(-1)}(d) \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} \ell\, T(m,\ell)(x^d)^m (y^d)^\ell. \qquad (8)$$

Equation (2) implies

$$\sum_{n,k\geq 1} T(n,k)k\, x^n y^{k-1} = \frac{\partial}{\partial y}\left(\frac{(1+xy-x^2)xy}{(1-x)(1-x^2-x^2y^2)}\right) = \frac{g(x,y)}{y}, \qquad (9)$$

where $g(x,y)$ is defined by equation (6). Equation (5) then follows from equations (8) and (9). □

References

[1] P. Hadjicostas and L. Zhang (2017), "Sommerville's symmetrical cyclic compositions of a positive integer with parts avoiding multiples of an integer," *Fibonacci Quart.*, **55**, 54–73.

[2] J. P. McSorley and A. H. Shoen (2013), "Rhombic tilings of (n, k)-ovals, (n, k, λ)-cyclic difference sets, and related topics," *Discrete Math.*, **313**, 129–154.

[3] D. M. Y. Sommerville (1909), "On Certain Periodic Properties of Cyclic Compositions of Numbers," *Proc. London Math. Soc.*, **S2–7**, 263–313.