Proof of a conjecture stated in A176328

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For $n \in \mathbb{N}_0$ let $a(n) = $ A176328(n). The statement in the following theorem was conjectured in A176328.

Theorem 1. *Let $n \in \mathbb{N}_0$. Then $a(2n+1) = -(2n+1)$.*

Proof. By a formula in A176328, for $m \in \mathbb{N}_0$, $a(m)$ is equal to the numerator of

$$\frac{(-1)^m(B_m(1) + B_m(2))}{2},$$

where $B_m(x)$ denotes the mth Bernoulli polynomial. It is well-known that

$$B_m(x+1) - B_m(x) = mx^{m-1}, \quad m \in \mathbb{N},$$
$$B_m(1) = 0, \quad m \geq 3 \text{ is odd},$$

Thus, if $n \geq 1$,

$$B_{2n+1}(2) = B_{2n+1}(2) - B_{2n+1}(1) = (2n+1) \cdot 1^{2n} = 2n+1.$$

It follows that

$$\frac{(-1)^{2n+1}(B_{2n+1}(1) + B_{2n+1}(2))}{2} = \frac{(-1)^{2n+1}(0 + 2n + 1)}{2} = -\frac{2n+1}{2},$$

as desired. For $n = 0$ we have $B_1(x) = x - \frac{1}{2}$ and therefore

$$\frac{(-1)^1(B_1(1) + B_1(2))}{2} = \frac{(-1)^1(1 - \frac{1}{2} + 2 - \frac{1}{2})}{2} = -1,$$

completing the proof. □

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.