Notations

We use the definition of $T(n, k) = \lceil \frac{n+1}{k} - \frac{k+1}{2} \rceil$ and $S(n, k) = T(n, k) - T(n, k+1) = \lceil \frac{n+1}{k} - \frac{k+1}{2} \rceil - \lceil \frac{n+1}{k+1} - \frac{k+2}{2} \rceil$, for $1 \le n$ and $1 \le k \le \lfloor \frac{1}{2} \left(\sqrt{8n+1} - 1 \right) \rfloor$ = row(n) from A237591 and A237593, respectively. The numbers S(n, k) and S(n-1, k), for a fixed n, are the lengths of the legs of one half of each the two symmetric paths p(n) and p(n-1) bounding the symmetric representation of σ(n).

Conjecture #1

The numbers n for which there is exactly one part in the symmetric representation of σ(n) and whose width is one unit square are exactly the powers of two, A000079.

Observations

Since T(n, 1) = n, for all $1 \le n$, the two paths p(n) and p(n-1) start one unit apart. Therefore, they stay exactly one unit apart when T(n, k) = T(n-1, k), for all $2 \le k \le$ row(n), because the respective pairs of legs of the two bounding paths have the same lengths.

Claims

(1) $T(2^n, k) = T(2^n - 1, k)$ holds for all $k, n \in \mathbb{N}$ with $2 \le k \le \text{row}(2^n)$.

(2) If for all $k, n \in \mathbb{N}$ with $2 \le k \le$ row(n), $T(n, k) = T(n - 1, k)$ holds, then $n = 2^m$ for suitable $m \in \mathbb{N}$.

Proof of Claim (1)

When k is odd, direct computation establishes $T(2^n, k) - T(2^n-1, k) = 0$.

When k is even, suppose that $\frac{2^n}{k} = s + \frac{d}{k}$ with $s, d \in \mathbb{N}$ and $0 \le d < k$.

Case $0 \le \frac{d}{k} < \frac{1}{2}$: then $s - 1 - \frac{k}{2} < \frac{2^n+1}{k} - \frac{k+1}{2} < s - \frac{k}{2}$ and $s - 1 - \frac{k}{2} < \frac{2^n}{k} - \frac{k+1}{2} \le s - \frac{1}{k} - \frac{k}{2}$.

Case $\frac{1}{2} < \frac{d}{k} < 1$: then $s - \frac{k}{2} < \frac{2^n+1}{k} - \frac{k+1}{2} < s + 1 - \frac{k}{2}$ and $s - \frac{k}{2} < \frac{2^n}{k} - \frac{k+1}{2} < s + 1 - \frac{k}{2}$.

Case $\frac{d}{k} = \frac{1}{2}$: cannot occur, since with k = 2m we get $\frac{2^n}{k} = s + \frac{1}{2}$, that is, $2^n = m\,(2s + 1)$.

Proof of Claim (2)

Suppose that $\lceil \frac{n+1}{k} - \frac{k+1}{2} \rceil = \lceil \frac{n}{k} - \frac{k+1}{2} \rceil$ holds for all $2 \le k \le \text{row}(n)$.

First observe that for $n \ge 6$ there is at least one odd $k \ge 3$ in the range stated above. Therefore, if k is odd, the equality above reduces to $\lceil \frac{n+1}{k} \rceil = \lceil \frac{n}{k} \rceil$ which implies that k cannot be a divisor of n.

Therefore, the only prime divisor of n less than or equal to row(n) is 2. Since $\sqrt{n} \le$ row(n) for all $n \in \mathbb{N}$, there are no other prime divisors, i.e., $n = 2^m$, for suitable $m \in \mathbb{N}$.

Conjecture #2

The numbers n for which there is exactly one part in the symmetric representation of σ(n) and whose width is one unit square except for two units at the midpoint are exactly the perfect numbers, A000396.

Observations

An even perfect number n has the form $(2^p-1)\,2^{p-1}$ where p is a prime and (2^p-1) is a Mersenne prime. Direct computation establishes the following three properties:

(a) $\text{row}\big((2^p-1)\,2^{p-1}\big) = 2^p - 1$

(b) every divisor of $(2^p-1)\,2^{p-1}$ less than $\text{row}\big((2^p-1)\,2^{p-1}\big)$ is a power of 2.

(c) $T((2^p-1)\,2^{p-1}, 2^p-1) = \left\lceil \frac{1}{2^p-1} \right\rceil = 1$ and $T[(2^p-1)\,2^{p-1}-1, 2^p-1] = 0$

Claims

(1) For all even perfect numbers $n = (2^p-1)\,2^{p-1}$, $T(n, k) - T(n-1, k) = 0$ holds for all $2 \le k < \text{row}(n)$.

(2) If for all $k, n \in \mathbb{N}$ such that $2 \le k < \text{row}(n)$,
$T(n, k) - T(n-1, k) = 0$ and $T(n, \text{row}(n)) - T(n-1, \text{row}(n)) = 1$
then n is an even perfect number.

The proofs follow the proof patterns for conjecture #1.

Proof of (1)

Let $\frac{n}{k} = \frac{(2^p-1)\,2^{p-1}}{k} = s + \frac{d}{k}$ with $s, d \in \mathbb{N}$, and note that $0 < d < k < 2^p - 1$ since $2^p - 1$ is a prime.

When k is odd, a direct computation establishes $T((2^p-1)\,2^{p-1}, k) - T((2^p-1)\,2^{p-1} - 1, k) = 0$.

When $k = 2m$ is even,

Case $0 \le \frac{d}{k} < \frac{1}{2}$:

$$T((2^p-1)\,2^{p-1} - 1, k) = \left\lceil \frac{(2^p-1)\,2^{p-1}}{k} - \frac{k+1}{2} \right\rceil = \lceil s + \frac{d}{k} - m - \frac{1}{2} \rceil = s - m + \lceil \frac{d}{k} - \frac{1}{2} \rceil = s - m$$

$$T((2^p-1)\,2^{p-1}, k) = \left\lceil \frac{(2^p-1)\,2^{p-1}+1}{k} - \frac{k+1}{2} \right\rceil = \lceil s + \frac{d+1}{k} - m - \frac{1}{2} \rceil = s - m + \lceil \frac{d+1}{k} - \frac{1}{2} \rceil = s - m$$

Case $\frac{1}{2} < \frac{d}{k} < 1$:

$$T((2^p-1)\,2^{p-1} - 1, k) = \left\lceil \frac{(2^p-1)\,2^{p-1}}{k} - \frac{k+1}{2} \right\rceil = \lceil s + \frac{d}{k} - m - \frac{1}{2} \rceil = s - m + \lceil \frac{d}{k} - \frac{1}{2} \rceil = s - m + 1$$

$$T((2^p-1)\,2^{p-1}, k) = \left\lceil \frac{(2^p-1)\,2^{p-1}+1}{k} - \frac{k+1}{2} \right\rceil = \lceil s + \frac{d+1}{k} - m - \frac{1}{2} \rceil = s - m + \lceil \frac{d+1}{k} - \frac{1}{2} \rceil = s - m + 1$$

CASE $\frac{1}{2} = \frac{d}{k}$:

This case cannot occur since $\frac{(2^p-1)\,2^{p-1}}{k} = s + \frac{1}{2}$ implies $(2^p-1)\,2^{p-1} = m\,(2s+1)$, that is, either $(2^p-1) \mid m$ or $(2^p-1) \mid (2s+1)$. The first is impossible since $2^p - 1 > m$.
The second implies $2s + 1 = q\,(2^p-1)$, for some odd $q \in \mathbb{N}$, which in turn implies $2^{p-1} = m\,q$. However, that is impossible since q is odd.

Proof of (2)

Suppose that $\left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil = \left\lceil \frac{n}{k} - \frac{k+1}{2} \right\rceil$ holds for all $2 \le k < \text{row}(n)$. First observe that for $n \ge 15$ there is at least one odd $k \ge 3$ in the range stated above. Therefore, if k is odd, the claimed equality reduces to $\left\lceil \frac{n+1}{k} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$ which, in turn, implies that k cannot be a divisor of n. Therefore, the onlyprime divisor of n less than row(n) is 2. Since by assumption $T(n, \text{row}(n)) - T(n-1, \text{row}(n)) = 1$, Conjecture (1) implies that n is not a power of 2, and since $\sqrt{n} \le \text{row}(n)$, for all $n \in \mathbb{N}$, the only other possible prime divisor of n is $x = \frac{1}{2}\left(\sqrt{8 n + 1} - 1\right)$, that is, $n = \frac{x}{2}(x + 1)$. Therefore, $\frac{x}{2}(x + 1) = n = x\, 2^y$, for suitable $y \in \mathbb{N}$ implies $x = 2^{y+1} - 1$. This shows $n = (2^{y+1} - 1)\, 2^y$, and since $2^{y+1} - 1$ must be a prime, n is a perfect number.