NUMBERS THAT ARE NEARLY DOUBLED WHEN REVERSED

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ABSTRACT. We find all numbers N such that $\operatorname{rev}(N) = 2N - 1$, where $\operatorname{rev}(N)$ is the digit reversal of N.

Digit reversal is a frequent topic in recreational mathematics. A number that is unchanged when its digits are reversed is called a *palindrome*. At the time of writing, there are 2366 sequences in the On-Line Encyclopedia of Integer Sequences that mention palindromes in their descriptions.

It is interesting to investigate the possible relations that can exist between a number and its digit reversal. For example, 2178 has the remarkable property that its reversal is four times as large. In fact, there are infinitely many numbers with this property.

$$2178 \times 4 = 8712$$
$$21978 \times 4 = 87912$$
$$219978 \times 4 = 879912$$
$$2199978 \times 4 = 8799912$$
$$\cdots$$

We will prove that there is no number (except 0) whose reversal is twice as large. However, there are infinitely many *near misses*; numbers N whose reversal is equal to $2N - 1$. We will prove that the sequence shown below represents all solutions to $\operatorname{rev}(N) = 2N - 1$.

$$1 \times 2 - 1 = 1$$
$$37 \times 2 - 1 = 73$$
$$397 \times 2 - 1 = 793$$
$$3997 \times 2 - 1 = 7993$$
$$39997 \times 2 - 1 = 79993$$
$$\cdots$$

Formally, the *digit reversal* of an n-digit number

$$N = \sum_{i=0}^{n-1} d_i 10^i, \qquad d_i \in \{0, 1, 2, \ldots, 9\}$$

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is defined by

$$\mathrm{rev}_n(N) = \sum_{i=0}^{n-1} d_{n-1-i} 10^i.$$

Leading zeros are allowed. For example, $\mathrm{rev}_3(123) = 321$, but $\mathrm{rev}_4(123) = 3210$.

The digit reversal of an n-digit number can be computed recursively, by reversing the first $(n-1)$ digits, then moving the last digit to the front. Formally,

$$\mathrm{rev}_n(10a + b) = 10^{n-1} b + \mathrm{rev}_{n-1}(a) \qquad (0 \le a < 10^{n-1}, 0 \le b \le 9)$$

with initial condition $\mathrm{rev}_0(0) = 0$.

Theorem 1. $\mathrm{rev}_n(\mathrm{rev}_n(N)) = N$ *for* $0 \le N < 10^n$.

Proof. Let $N = \sum_{i=0}^{n-1} d_i 10^i$, where $d_i \in \{0, 1, 2, \ldots, 9\}$. Then

$$\begin{aligned}
\mathrm{rev}_n(\mathrm{rev}_n(N)) &= \mathrm{rev}_n\left(\sum_{i=0}^{n-1} d_i 10^i\right) \\
&= \mathrm{rev}_n\left(\sum_{i=0}^{n-1} d_{n-1-i} 10^i\right) \\
&= \sum_{i=0}^{n-1} d_{n-1-(n-1-i)} 10^i \\
&= \sum_{i=0}^{n-1} d_i 10^i \\
&= N.
\end{aligned}$$

□

Theorem 2. *If* $S = 10^n - 1$ *and* $0 \le N \le S$ *then* $\mathrm{rev}_n(S - N) = S - \mathrm{rev}_n(N)$.

Proof. Let $N = \sum_{i=0}^{n-1} d_i 10^i$, and note that $S = \sum_{i=0}^{n-1} 9 \cdot 10^i$. Therefore

$$\begin{aligned}
\mathrm{rev}_n(S - N) &= \mathrm{rev}_n\left(\sum_{i=0}^{n-1} 9 \cdot 10^i - \sum_{i=0}^{n-1} d_i 10^i\right) \\
&= \mathrm{rev}_n\left(\sum_{i=0}^{n-1} (9 - d_i) \cdot 10^i\right) \\
&= \sum_{i=0}^{n-1} (9 - d_{n-1-i}) \cdot 10^i \\
&= \sum_{i=0}^{n-1} 9 \cdot 10^i - \sum_{i=0}^{n-1} d_{n-1-i} 10^i \\
&= S - \mathrm{rev}_n(N).
\end{aligned}$$

□

The reader is familiar with the standard algorithm for adding positive integers, but we will give a formal description. Let A and B be nonnegative integers whose decimal expansions are $A = \sum_i a_i 10^i$ and $B = \sum_i b_i 10^i$ respectively. Suppose that the sum $A + B$ has decimal expansion $A + B = \sum_i d_i 10^i$. Then

$$a_i + b_i + c_{i-1} = d_i + 10c_i \tag{1}$$

where $c_{-1} = 0$ and $c_i \in \{0, 1\}$ depending on whether a carry occurred in the column for 10^i.

If $A = B$ then the equation can be written as

$$2a_i + c_{i-1} = d_i + 10c_i. \tag{2}$$

Theorem 3. *The only solution to* $\operatorname{rev}_n(N) = 2N$ *with* $0 \le N < 10^n$ *is* $N = 0$.

Proof. The proof is by induction on n. The case $n = 0$ is trivial.

Let $N = \sum_{i=0}^{n-1} a_i 10^i$ where $a_i \in \{0, 1, 2, \ldots, 9\}$, and suppose that $\operatorname{rev}_n(N) = 2N$. By equation 2,

$$2a_0 = a_{n-1} + 10c_0$$

and

$$2a_{n-1} + c_{n-2} = a_0.$$

Solving this system for a_0 and a_{n-1} yields

$$a_0 = (20c_0 - c_{n-2})/3$$

and

$$a_{n-1} = (10c_0 - 2c_{n-2})/3.$$

Since $c_0 \in \{0, 1\}$ and $c_{n-2} \in \{0, 1\}$, one obtains integer values for a_0 and a_{n-1} only when $c_0 = c_{n-2} = 0$, hence $a_0 = a_{n-1} = 0$. Since the first and last digits of N are zeros, it follows that

$$\operatorname{rev}_{n-2}(N/10) = 2(N/10).$$

By the induction hypothesis, $N/10 = 0$, hence $N = 0$. □

Theorem 4. *If* $N = 4 \cdot 10^{n-1} - 3$ *then* $\operatorname{rev}_n(N) = 2N - 1$.

Proof. The cases $n = 1$ and $n = 2$ are easy to check, so we suppose that $n \ge 3$. The decimal expansion of

$$N = 4 \cdot 10^{n-1} - 3$$

is

$$N = 3999...97$$

with $n - 2$ nines. But

$$2N - 1 = 8 \cdot 10^{n-1} - 7$$

and its decimal expansion is

$$2N - 1 = 7999...93$$

with $n-2$ nines. Therefore, $\mathrm{rev}_n(N) = 2N-1$. □

Theorem 5. *If* $\mathrm{rev}_n(N) = 2N-1$ *and* $0 \leq N < 10^n$ *then* $N = 4\cdot 10^{n-1} - 3$.

Proof. This can be verified for $n \leq 2$ by brute force, so let us assume that $n \geq 3$.

Let $N = \sum_{i=0}^{n-1} a_i 10^i$ where $a_i \in \{0,1,2,\ldots,9\}$, and suppose that $\mathrm{rev}_n(N) = 2N-1$. If $a_0 = 0$ then $a_{n-1} = 9$; but this is impossible, since $\mathrm{rev}_n(N) > N$.

Since $a_0 \geq 1$, equation 2 implies that

$$2a_0 - 1 = a_{n-1} + 10c_0$$

and

$$2a_{n-1} + c_{n-2} = a_0.$$

Solving this system for a_0 and a_{n-1} yields

$$a_0 = (20c_0 - c_{n-2} + 2)/3$$

and

$$a_{n-1} = (10c_0 - 2c_{n-2} + 1)/3.$$

One obtains integer values for a_0 and a_{n-1} only when $c_0 = c_{n-2} = 1$, hence $a_0 = 7$ and $a_{n-1} = 3$. Therefore, there exists an integer A such that $0 \leq A < 10^{n-2}$ and

$$N = 3\cdot 10^{n-1} + 10A + 7.$$

Since $\mathrm{rev}_n(N) = 2N-1$, it follows that

$$2N - 1 = 7\cdot 10^{n-1} + 10B + 3 \tag{3}$$

where $B = \mathrm{rev}_{n-2}(A)$.

Eliminating N from the two equations yields

$$0 = -10^{n-1} + 20A - 10B + 10$$

which can be rewritten as

$$2(S - A) = (S - B)$$

where $S = 10^{n-2} - 1$.

By Theorem 2,

$$S - B = S - \mathrm{rev}_{n-2}(A) = \mathrm{rev}_{n-2}(S - A)$$

so Theorem 3 implies that $S - A = 0$. Therefore (by equation 3)

$$N = 3\cdot 10^{n-1} + 10(10^{n-2} - 1) + 7 = 4\cdot 10^{n-1} - 3.$$

□