A NEW INTERPRETATION OF HULTMAN NUMBERS

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Abstract. In this short note, we provide a new interpretation of Hultman numbers $S_H(2n-2,1)$ in OEIS "A164652".

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1. Introduction

In this short note, we provide a new interpretation of Hultman numbers $S_H(2n-2,1)$. $S_H(2n-2,1)$ denotes the number of spanning trees of an infinite series of graphs.

We recall Hultman numbers $S_H(n,k)$ [2, 1]. Let S_n be the permutation group with respect to $\{1,2,\ldots,n\}$. For $\pi \in S_n$, $\pi_i = \pi(i)$.

Definition 1.1 ([2, Definitions 3 and 4])**.** The cycle graph of a permutation $\pi \in S_n$ is the bicolored directed graph $G(\pi)$ with vertex set $\{\pi_0 = 0, \pi_1, \ldots, \pi_n\}$ and its edge set consists of

- black edges $(\pi_i, \pi(i_1) \pmod{n+1})$ for $0 \le i \le n$, and
- grey edges $(i, (i+1) \pmod{n+1})$ for $0 \le i \le n$.

The Hultman number $S_H(n,k)$ counts the number of permutations in S_n whose cycle graph decomposes into k alternating cycles. Thus,

$$S_H(n,k) = |\{\pi \in S_n \mid c(G(\pi)) = k\}|,$$

where $c(G(\pi))$ is the number of cycles in $G(\pi)$.

Let $G_n = (V_n, E_n)$ be a simple graph that satisfies the following conditions:

- $V_n = \{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, \ldots, a_{n,1}, a_{n,2}\}$.
- For all $\ell \in \{1,2,\ldots,n\}$, there is no edge between $a_{\ell,1}$ and $a_{\ell,2}$.
- The set of degrees is $\{0,1,\ldots,n-1,n-1,n,\ldots,2n-2\}$.

It is easy to see that G_n is uniquely determined by these conditions. (See Appendix A.)

The main result of this short note is as follows:

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Theorem 1.1. *Let $G_n \setminus v_0$ be a graph in which the degree 0 vertex has been deleted and $\tau(G_n \setminus v_0)$ is the number of spanning trees of $G_n \setminus v_0$. Then we have*

$$\tau(G_n \setminus v_0) = \frac{(2(n-1))!}{n} = S_H(2n-2,1).$$

(See also Appendix B and C.)

In the next section, we provide a proof of Theorem 1.1.

2. Proof of Theorem 1.1

Let v_1 be the degree one vertex and $\widetilde{G}_n$ be the graph in which v_1 has been deleted and the edge adjacent to v_1 from $G_n \setminus v_0$ has been deleted. It is sufficient to show that the number of spanning trees of $\widetilde{G}_n$ is $S_H(2n-2,1)$.

We note that there exist two degree $2n-3$ vertices. Let $\widetilde{L}_n$ be a matrix in which the row indexed by the degree $2n-3$ vertex and the column indexed by another degree $2n-3$ vertex have been deleted from the Laplacian matrix of $\widetilde{G}_n$. $\widetilde{L}_n$ can be written as follows:

$$\widetilde{L}_n = \begin{bmatrix} -1 & -1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots \\ * & 2n-4 & 0 & -1 & -1 & \cdots & \cdots & \cdots \\ * & * & 2 & 0 & 0 & 0 & \cdots & \cdots \\ * & * & * & 2n-5 & 0 & -1 & -1 & \cdots \\ * & * & * & * & 3 & 0 & 0 & \cdots \\ * & * & * & * & * & \ddots & \ddots & \cdots \\ * & * & * & * & * & * & n-1 & 0 \\ * & * & * & * & * & * & * & n-1 \end{bmatrix}.$$

By the matrix tree theorem, it is sufficient to compute the determinant of $\widetilde{L}_n$.

In fact, by checking the direct computations, we have the following complete list of eigenvalues and their eigenvectors of $\widetilde{L}_n$:

Eigenvalue	Eigenvector
$(1+\sqrt{8n-7})/2$	$[(3-\sqrt{8n-7})/2, 1, \ldots, 1]$
$(1-\sqrt{8n-7})/2$	$[(3+\sqrt{8n-7})/2, 1, \ldots, 1]$
$2n-3$	$[0, -(2n-6), 0, 1, 1, \ldots, 1]$
$2n-4$	$[0, 0, 0, -(2n-8), 0, 1, 1, \ldots, 1]$
$\vdots$	$\vdots$
$n+1$	$[0, 0, \ldots, 0, -2, 0, 1, 1]$
$n-1$	$[0, 0, \ldots, 0, 0, -1, 1]$
$n-2$	$[0, 0, \ldots, 0, 1, -3, 1, 1]$
$n-3$	$[0, 0, \ldots, 0, 1, -5, 1, 1, 1]$
$\vdots$	$\vdots$
2	$[0, 1, -(2n-5), 1, 1 \ldots, 1]$

Then by the matrix tree theorem,

$$\begin{aligned}\tau(G_n \setminus v_0) =& \mid \det(\widetilde{L}_n) \mid \\ =& -\frac{1}{2}(1+\sqrt{8n-7})\frac{1}{2}(1-\sqrt{8n-7}) \\ & (2n-3)(2n-4)\cdots(n+1) \\ & (n-1)(n-2)\cdots 2 \\ =& \frac{(2(n-1))!}{n} = S_H(2n-2,1).\end{aligned}$$

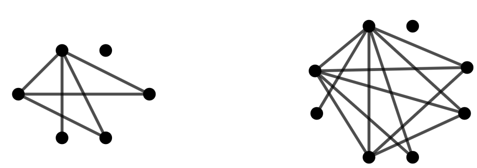
This completes the proof of Theorem 1.1.

Remark 2.1. (1) Is there a bijective proof of Theorem 1.1; that is, can we construct a bijection between $\{\pi \in S_{2n-2} \mid c(G(\pi)) = 1\}$ and the set of spanning trees of $G_n \setminus v_0$?

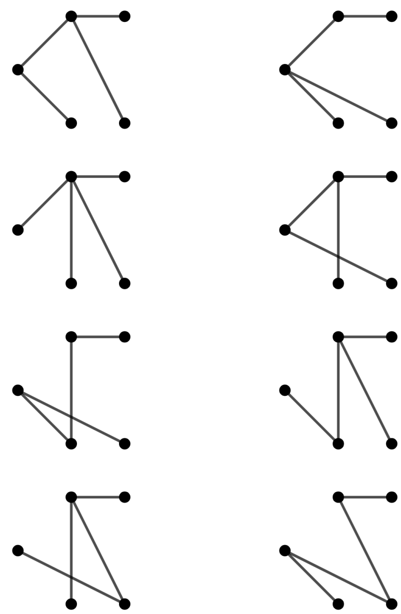
(2) For all n, k, is $S_H(n,k)$ the number of spanning trees of some infinite series of graphs $G_{n,k}$?

Appendix A. G_3 and G_4

G_3 G_4



Appendix B. Spanning trees of $G_3 \setminus v_0$



Appendix C. Elements of $\{\pi \in S_4 \mid c(G(\pi)) = 1\}$

(2143) | (2413) | (2431) | (3142) | (3241) | (4132) | (4213) | (4321)

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