How_I_found_A157852_sum.nb

Mission: integrate the terms of the Taylor series and find a pattern for their sum.

In[∘]:=
```
l2 = Normal[Series[Exp[I Pi x] (x^(1/x) - 1), {x, Infinity, 3}]];
```

In[∘]:=
```
l3 = NIntegrate[l2, {x, 1, Infinity I}]
```

Out[∘]=
```
0.070776 - 0.0473806 i
```

In[∘]:=
```
l2
```

Out[∘]=

$$e^{i \pi x}\left(\frac{\text{Log}[x]}{x}+\frac{\text{Log}[x]^2}{2 x^2}+\frac{\text{Log}[x]^3}{6 x^3}\right)$$

In[∘]:=
```
Integrate[l2, {x, 1, Infinity I}]
```

Out[∘]=
```
MeijerG[{{}, {1, 1}}, {{0, 0, 0}, {}}, -i π] -
 i π MeijerG[{{}, {1, 1, 1}}, {{-1, 0, 0, 0}, {}}, -i π] -
 π^2 MeijerG[{{}, {1, 1, 1, 1}}, {{-2, 0, 0, 0, 0}, {}}, -i π]
```

The above is the pattern for the sum. What does it look like?

In[∘]:=
```
l2 = Normal[Series[Exp[I Pi x] (x^(1/x) - 1), {x, Infinity, 3}]];
```

The following are the same!

$$\int_1^{\infty}(x^{1/x}-1)\exp(i\,\pi\,x)\,dx=\int_1^{i\,\infty}(x^{1/x}-1)\exp(i\,\pi\,x)\,dx$$

In[∘]:=
```
FullSimplify[N[NIntegrate[Exp[I Pi x] (x^(1/x) - 1),
     {x, 1, Infinity}, WorkingPrecision → 160, MaxRecursion → 300], 60] -
   NIntegrate[Exp[I Pi x] (x^(1/x) - 1), {x, 1, Infinity I}, WorkingPrecision → 60]]
```

Out[∘]=
```
0. × 10^-61 + 0. × 10^-61 i
```

In[∘]:=
```
NIntegrate[l2, {x, 1, Infinity }] - NIntegrate[l2, {x, 1, Infinity I}]
```

NIntegrate: DoubleExponentialOscillatory returns a finite integral estimate, but the integral might be divergent.

NIntegrate: DoubleExponentialOscillatory returns a finite integral estimate, but the integral might be divergent.

Out[∘]= $-1.04802\times10^{-11} - 1.62902\times10^{-9}\,i$

In[∘]:=
```
l2
```

Out[∘]= $e^{i\pi x}\left(\frac{\text{Log}[x]}{x} + \frac{\text{Log}[x]^2}{2x^2} + \frac{\text{Log}[x]^3}{6x^3}\right)$

In[∘]:=
```
Integrate[l2, {x, 1, Infinity}]
```

Out[∘]=

$$
\begin{aligned}
&\frac{1}{3840}\times\Big(-80\,\text{EulerGamma}^4\,\pi^2+3\,\pi^6+\\
&\quad 160\,\text{EulerGamma}^3\,\pi\,\left(-4\,i+i\,\pi^2+\pi\,(3-2\,\text{Log}[\pi])\right)+20\,i\,\pi^5\,(-3+2\,\text{Log}[\pi])+10\,\pi^4\\
&\quad\left(6+\text{HypergeometricPFQ}\left[\{1,1,1,1,1\},\left\{2,2,2,2,\tfrac{5}{2},3\right\},-\tfrac{\pi^2}{4}\right]-12\,\text{Log}[\pi]+4\,\text{Log}[\pi]^2\right)+\\
&\quad 240\times\left(1+16\,\text{HypergeometricPFQ}\left[\left\{-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right\},\left\{\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right\},-\tfrac{\pi^2}{4}\right]+8\,\text{Log}[\pi]^2\right)+\\
&\quad 40\,\text{EulerGamma}^2\\
&\quad\left(48+\pi^4-48\,i\,\pi\,(-1+\text{Log}[\pi])+6\,i\,\pi^3\,(-3+2\,\text{Log}[\pi])-6\,\pi^2\left(11-6\,\text{Log}[\pi]+2\,\text{Log}[\pi]^2\right)\right)+\\
&\quad 640\,i\,\pi\,\Big(6+6\,\text{HypergeometricPFQ}\left[\left\{\tfrac{1}{2},\tfrac{1}{2}\right\},\left\{\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}\right\},-\tfrac{\pi^2}{4}\right]+\\
&\qquad 6\,\text{HypergeometricPFQ}\left[\left\{-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right\},\left\{\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2}\right\},-\tfrac{\pi^2}{4}\right]-\\
&\qquad 9\,\text{Log}[\pi]+3\,\text{Log}[\pi]^2-\text{Log}[\pi]^3-2\,\text{Zeta}[3]\Big)+\\
&\quad 40\,i\,\pi^3\,\Big(-49+2\,\text{HypergeometricPFQ}\left[\{1,1,1,1\},\left\{2,2,2,2,\tfrac{5}{2}\right\},-\tfrac{\pi^2}{4}\right]+\\
&\qquad 46\,\text{Log}[\pi]-18\,\text{Log}[\pi]^2+4\,\text{Log}[\pi]^3+8\,\text{Zeta}[3]\Big)-\\
&\quad 40\,\pi^2\,\Big(145+12\,\text{HypergeometricPFQ}\left[\{1,1,1\},\left\{\tfrac{3}{2},2,2,2\right\},-\tfrac{\pi^2}{4}\right]+66\,\text{Log}[\pi]^2-\\
&\qquad 12\,\text{Log}[\pi]^3+2\,\text{Log}[\pi]^4-24\,\text{Zeta}[3]+2\,\text{Log}[\pi]\,(-69+8\,\text{Zeta}[3])\Big)+\\
&\quad 40\,\text{EulerGamma}\,\Big(i\,\pi^5+96\,\text{Log}[\pi]+\pi^4\,(-3+2\,\text{Log}[\pi])-48\,i\,\pi\left(3-2\,\text{Log}[\pi]+\text{Log}[\pi]^2\right)+2\,i\,\pi^3\\
&\qquad\left(23-18\,\text{Log}[\pi]+6\,\text{Log}[\pi]^2\right)-2\,\pi^2\left(-69+66\,\text{Log}[\pi]-18\,\text{Log}[\pi]^2+4\,\text{Log}[\pi]^3+8\,\text{Zeta}[3]\right)\Big)\Big)
\end{aligned}
$$

That is too messy!

In[∘]:=
```
l2 = Normal[Series[Exp[I Pi x] (x^(1/x) - 1), {x, Infinity, 6}]];
```

```
In[•]:= l3 = NIntegrate[l2, {x, 1, Infinity I}]

Out[•]= 0.070776 - 0.0473807 ⅈ

In[•]:= l2
```

Out[•]= $e^{i\pi x}\left(\frac{\text{Log}[x]}{x}+\frac{\text{Log}[x]^2}{2x^2}+\frac{\text{Log}[x]^3}{6x^3}+\frac{\text{Log}[x]^4}{24x^4}+\frac{\text{Log}[x]^5}{120x^5}+\frac{\text{Log}[x]^6}{720x^6}\right)$

```
In[•]:= Integrate[l2, {x, 1, Infinity I}]

Out[•]= MeijerG[{{}, {1, 1}}, {{0, 0, 0}, {}}, -ⅈ π] -
  ⅈ π MeijerG[{{}, {1, 1, 1}}, {{-1, 0, 0, 0}, {}}, -ⅈ π] -
  π^2 MeijerG[{{}, {1, 1, 1, 1}}, {{-2, 0, 0, 0, 0}, {}}, -ⅈ π] +
  ⅈ π^3 MeijerG[{{}, {1, 1, 1, 1, 1}}, {{-3, 0, 0, 0, 0, 0}, {}}, -ⅈ π] +
  π^4 MeijerG[{{}, {1, 1, 1, 1, 1, 1}}, {{-4, 0, 0, 0, 0, 0, 0}, {}}, -ⅈ π] -
  ⅈ π^5 MeijerG[{{}, {1, 1, 1, 1, 1, 1, 1}}, {{-5, 0, 0, 0, 0, 0, 0, 0}, {}}, -ⅈ π]
```

Above is more of the pattern.

```
MeijerG[{{}, {1, 1}}, {{0, 0, 0}, {}}, -ⅈ π] -
 ⅈ π MeijerG[{{}, {1, 1, 1}}, {{-1, 0, 0, 0}, {}}, -ⅈ π] -
 π^2 MeijerG[{{}, {1, 1, 1, 1}}, {{-2, 0, 0, 0, 0}, {}}, -ⅈ π]
The above is the pattern for the sumand.What does it look like?
    MeijerG[{{}, {1, 1}}, {{0, 0, 0}, {}}, -ⅈ π] -
 ⅈ π MeijerG[{{}, {1, 1, 1}}, {{-1, 0, 0, 0}, {}}, -ⅈ π] -
 π^2 MeijerG[{{}, {1, 1, 1, 1}}, {{-2, 0, 0, 0, 0}, {}}, -ⅈ π] +
 ⅈ π^3 MeijerG[{{}, {1, 1, 1, 1, 1}}, {{-3, 0, 0, 0, 0, 0}, {}}, -ⅈ π] +
 π^4 MeijerG[{{}, {1, 1, 1, 1, 1, 1}}, {{-4, 0, 0, 0, 0, 0, 0}, {}}, -ⅈ π] -
 ⅈ π^5 MeijerG[{{}, {1, 1, 1, 1, 1, 1, 1}}, {{-5, 0, 0, 0, 0, 0, 0, 0}, {}}, -ⅈ π]
There is more of the pattern.

In[•]:= N[MeijerG[{{}, {1, 1}}, {{0, 0, 0}, {}}, -I π] -
   I π MeijerG[{{}, {1, 1, 1}}, {{-1, 0, 0, 0}, {}}, -I π] -
   π^2 MeijerG[{{}, {1, 1, 1, 1}}, {{-2, 0, 0, 0, 0}, {}}, -I π] +
   I π^3 MeijerG[{{}, {1, 1, 1, 1, 1}}, {{-3, 0, 0, 0, 0, 0}, {}}, -I π] +
   π^4 MeijerG[{{}, {1, 1, 1, 1, 1, 1}}, {{-4, 0, 0, 0, 0, 0, 0}, {}}, -I π] -
   I π^5 MeijerG[{{}, {1, 1, 1, 1, 1, 1, 1}}, {{-5, 0, 0, 0, 0, 0, 0, 0}, {}}, -I π]]

Out[•]= 0.070776 - 0.0473807 ⅈ
```

Trying to get traditional form

In[*]:=
```
f[n_] := MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -i π];
```

Where

```
MeijerG[{{}, Table[1, {6 + 1}]}, {Prepend[Table[0, 6 + 1], -6 + 1], {}}, -i π];
```

```
-i π^5 MeijerG[{{}, {1, 1, 1, 1, 1, 1, 1}}, {{-5, 0, 0, 0, 0, 0, 0, 0}, {}}, -i π]
```

is shown as

$$i\,\pi^5\,G_{7,8}^{8,0}\left(-i\,\pi \,\middle|\, \begin{matrix} 1,1,1,1,1,1,1 \\ -5,0,0,0,0,0,0,0 \end{matrix}\right).$$

$$i\,\pi^5\,G_{7,8}^{8,0}\left(-i\,\pi \,\middle|\, \begin{matrix} 1,1,1,1,1,1,1 \\ -5,0,0,0,0,0,0,0 \end{matrix}\right).$$

In[*]:=
$$\sum_{n=1}^{6} \left(i^{1-n}\,\pi^{n-1}\right) f[n]$$

Out[*]=
```
MeijerG[{{}, {1, 1}}, {{0, 0, 0}, {}}, -i π] -
 i π MeijerG[{{}, {1, 1, 1}}, {{-1, 0, 0, 0}, {}}, -i π] -
 π^2 MeijerG[{{}, {1, 1, 1, 1}}, {{-2, 0, 0, 0, 0}, {}}, -i π] +
 i π^3 MeijerG[{{}, {1, 1, 1, 1, 1}}, {{-3, 0, 0, 0, 0, 0}, {}}, -i π] +
 π^4 MeijerG[{{}, {1, 1, 1, 1, 1, 1}}, {{-4, 0, 0, 0, 0, 0, 0}, {}}, -i π] -
 i π^5 MeijerG[{{}, {1, 1, 1, 1, 1, 1, 1}}, {{-5, 0, 0, 0, 0, 0, 0, 0}, {}}, -i π]
```

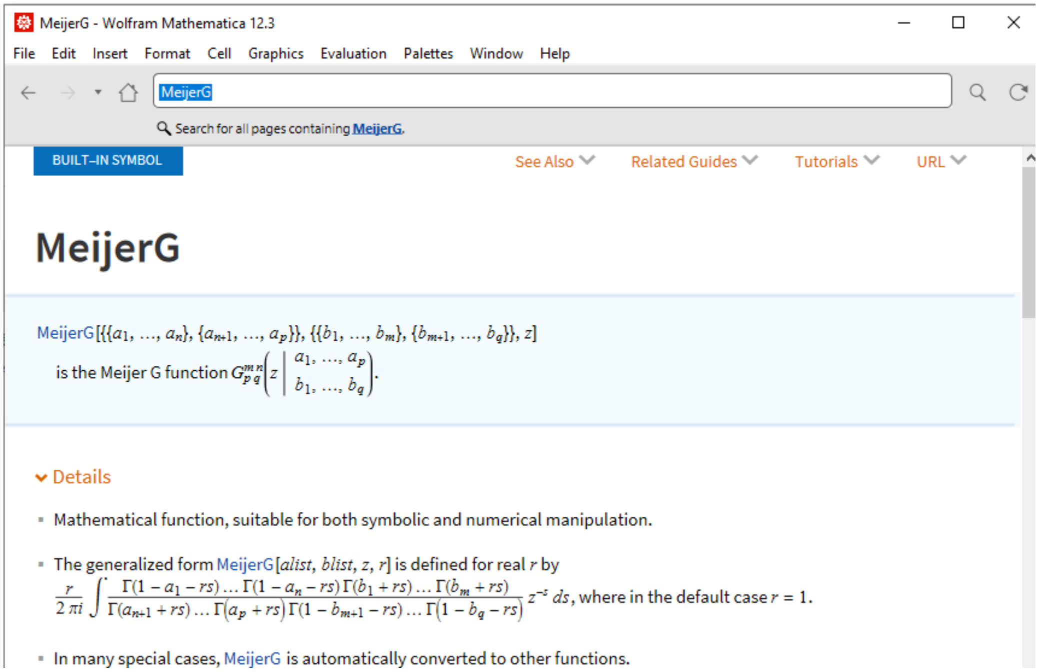
$$\sum_{n=1}^{6} \left(i^{1-n}\,\pi^{n-1}\right) f[n]$$

In[*]:=
```
N[%]
```

Out[*]=
```
0.070776 - 0.0473807 i
```

$$\sum_{n=1}^{6} (i\pi)^n \,\text{MeijerG}[\{\{\}, \{1, 1\}\}, \{\{n, n, n\}, \{\}\}, -i\,\pi]$$

$$i\,\pi^5\,G_{7,8}^{8,0}\left(-i\,\pi \,\middle|\, \begin{matrix} 1,1,1,1,1,1,1 \\ -5,0,0,0,0,0,0,0 \end{matrix}\right)$$



MeijerG - Wolfram Mathematica 12.3

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MeijerG

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MeijerG

MeijerG[{{a_1, …, a_n}, {a_{n+1}, …, a_p}}, {{b_1, …, b_m}, {b_{m+1}, …, b_q}}, z]

is the Meijer G function $G_{p\,q}^{m\,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$.

Details

- Mathematical function, suitable for both symbolic and numerical manipulation.
- The generalized form MeijerG[*alist*, *blist*, *z*, *r*] is defined for real *r* by $\frac{r}{2\pi i}\int \frac{\Gamma(1-a_1-rs)\ldots\Gamma(1-a_n-rs)\Gamma(b_1+rs)\ldots\Gamma(b_m+rs)}{\Gamma(a_{n+1}+rs)\ldots\Gamma(a_p+rs)\Gamma(1-b_{m+1}-rs)\ldots\Gamma(1-b_q-rs)} z^{-s}\, ds$, where in the default case $r = 1$.
- In many special cases, MeijerG is automatically converted to other functions.

The MRB constant

$$\text{CMRB} = \sum_{n=1}^{\infty} (-1)^n g(n) == -\text{Im}\left(\int_1^{i\infty} \frac{x^{1/x}-1}{\sin(\pi x)}\, dx\right)$$

$/.g(n)=(n^{1/n}-1)$.

The MRB constant's integrated analog

$$\text{M2} = \int_1^{\infty} (x^{1/x}-1)\, e^{i\pi x}\, dx = \; == \sum_{n=1}^{\infty} \left(\frac{i}{\pi}\right)^{1-n} f(n)$$

$/.\ f(n) = \text{MeijerG}[\{\{\}, \{1, 1, \ \ldots 1+n \text{ times}\}\}, \{\{1-n, 0, 0, \ \ldots 1+n \text{ times}\}, \{\}\}, -i\pi]$.

$$/.\ f(n) = G_{n+1\ \ n+2}^{n+2\ \ 0}\left(-i\pi \,\middle|\, \begin{matrix} 1, \overset{\text{nX}}{\cdots} \\ -n+1, 0, \underset{\text{nX}}{\ldots} \end{matrix}\right)$$

```
In[]:= f[n_] := MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π];
```

In[∘]:=
```
Sum[(ⅈ/π)^(1-n) f[n], {n, 1, Infinity}]
```

Table: Iterator {1 + n} does not have appropriate bounds.

Table: Iterator {1 + n} does not have appropriate bounds.

Table: Non-list iterator n + 1 at position 2 does not evaluate to a real numeric value.

Table: Non-list iterator n + 1 at position 3 does not evaluate to a real numeric value.

MeijerG: MeijerG[{{}, Table[1, {1 + n}]}, {Table[1 − n, 0, n + 1], {}}, −ⅈ π] does not exist. Arguments are not consistent.

Table: Iterator {1 + n} does not have appropriate bounds.

General: Further output of Table::iterb will be suppressed during this calculation.

Table: Non-list iterator n + 1 at position 3 does not evaluate to a real numeric value.

General: Further output of Table::nliter will be suppressed during this calculation.

MeijerG: MeijerG[{{}, Table[1, {1 + n}]}, {Table[1 − n, 0, n + 1], {}}, −ⅈ π] does not exist. Arguments are not consistent.

Out[∘]=
$$\sum_{n=1}^{\infty} \left(\frac{i}{\pi}\right)^{1-n} \text{MeijerG}[\{\{\}, \text{Table}[1, \{1+n\}]\}, \{\text{Table}[1-n, 0, n+1], \{\}\}, -i\pi]$$

Testing

In[∘]:=
```
f[n_] := MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π];
```

In[∘]:=
```
M2 = NIntegrate[E^(I Pi x) (x^(1/x) - 1), {x, 1, Infinity I}, WorkingPrecision → 100]
```

Out[∘]=
```
0.07077603931152880353952802183028200136575469620336302758317278816361845726438203658083
  188126617723821 -
  0.04738061707035078610720940650260367857315289969317363933196100090256586758807049779050
  462314770913485 ⅈ
```

In[∘]:=
```
A157852 = M2 - 2 I / Pi
```

Out[∘]=
```
0.07077603931152880353952802183028200136575469620336302758317278816361845726438203658083
  188126617723821 -
  0.68400038943793212918274445999266112671099148265499943432263037713815305812497663815095
  983421272147867 ⅈ
```

In[∘]:=
```
f[n_] := N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 30];
```

In[∘]:=
```
Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 10}] - 2 I / Pi
```

Out[∘]= 0.0707760393115461182041741292508 − 0.6840003894380241257856365790945 ⅈ

In[∘]:=
```
Abs[%]
```

Out[∘]= 0.6876523689277876599345116734241

```
f[n_] :=
 N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 60];
A157852 - Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 20}] - 2 I / Pi
```

Out[∘]= $Aborted

```
f[n_] :=
 N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 40];
A157852 - Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 20}] - 2 I / Pi
```

Out[∘]= $Aborted

```
f[n_] :=
 N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 30];
A157852 - Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 20}] - 2 I / Pi
```

Out[∘]= $Aborted

In[∘]:=
```
f[n_] := N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 30];
A157852 - Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 15}] + 2 I / Pi
```

Out[∘]= $-7.004855743 \times 10^{-22} + 1.2673662687 \times 10^{-21}\, i$

In[∘]:=
```
f[n_] :=
 N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 30];
M2 + 2 I / Pi - Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 16}] - 2 I / Pi
```

Out[∘]= $Aborted

```
In[ ]:= f[n_] := N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 22];
        A157852 - Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 15}] + 2 I / Pi

Out[ ]= -7.0 × 10^-22 + 1.27 × 10^-21 ⅈ

In[ ]:= Abs[A157852]

Out[ ]= 0.687652368927694369809312409365440164939637384903622541795071010107433662534784937068627
        29824049846819

In[ ]:= f[n_] := N[MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π], 22];
        Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 15}] - 2 I / Pi

Out[ ]= 0.07077603931152880354023 - 0.68400038943793212918401 ⅈ
```

```
        f[n_] := MeijerG[{{}, Table[1, {n + 1}]}, {Prepend[Table[0, n + 1], -n + 1], {}}, -ⅈ π];

In[ ]:= M2 - N[Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 6}], 30]

Out[ ]= 9.7363265318811615798137 × 10^-9 + 3.32301202826282224580827 × 10^-8 ⅈ

In[ ]:= M2 - N[Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 10}], 30]

Out[ ]= -1.73146646461074205 × 10^-14 + 9.19966028921191019 × 10^-14 ⅈ

In[ ]:= Timing[M2 - N[Sum[(ⅈ/π)^(1-n) f[n], {n, 1, 15}], 30]]

Out[ ]= {237.734, -7.004855743 × 10^-22 + 1.2673662687 × 10^-21 ⅈ}
```

The following took several hours .

```
In[6] := M2 - N[Sum[(I/π)^(1-n) f[n], {n, 1, 20}], 30]

Out[6] = -2.9 * 10^-30 + 3.6 * 10^-30 I
```