Derivation of recurrence relation for bisymmetric binary square matrices with row and column sum fixed to 2

January 26, 2025

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# 1 Introduction

I am not a professional mathematician, but a physicist, so my standard of rigour and presentation may differ from what an academic may expect. I derived this when solving Project Euler problem 741, and my post there contains an equivalent presentation. (This document is based off that.)

A key property I will use is that of *permutation invariance*. Namely, by permuting rows and columns of the considered matrices, the row and column sum stays invariant.

I found the argument deriving a recurrence for the unrestricted form of this problem very helpful Tan, Z., Gao, S. & Heinrich, N. Enumeration of $(0, 1)$-matrices with constant row and column sums. *Appl. Math. Chin. Univ.* **21**, 479–486 (2006).

Note, by bisymmetric I mean a symmetric matrix that is also symmetric about reflections of its anti-diagonal. A binary matrix is a matrix whose entries can only be 0 or 1.

# 2 Derivation

We will follow the strategy of peeling a square onion, so we shall divide the cases into even and odd dimension $n$. This is because fixing the first row also fixes the first and last rows and columns, by symmetry. Let $a_{2n}$ by the number of $(2n) \times (2n)$ binary square matrices with row and column sum fixed to 2. Let the number of bisymmetric matrices of the same dimensions with the first row like $11000 \cdots 0$ (one on corner, one next to that corner) be $b_{2(n-1)}$, and let the number of bisymmetric matrices whose first row is like $011000 \cdots 0$ be $c_{2(n-1)}$. Note we have the $-1$ because the outer square ring is fixed, so the free part has dimension $(2n-2) \times (2n-2)$.

First consider if the first row has 1s on both corners. By symmetry, all corners will have 1, whence we can delete the outer square onion. This contribution is $a_{2n-2}$. Now consider the case when the first row is symmetric under 180-degree rotation. Then by symmetry, the last row will be identical. Furthermore, the first and last column will be identical. Thus we have, in addition to the outer square onion, 2 extra rows and columns that are filled and can be deleted. The number of such first rows with the 180-deg symmetry, which do not have corner 1s is $(n-1)$. Thus, this contribution is $(n-1)a_{2n-4}$.

Now consider if the first row has a corner 1 and a non-corner 1. By permutation invariance, we can map this to the case of $11000 \cdots 0$, namely $b_{2(n-1)}$. The corner 1 has two options, whereas the non-corner has $(2n-2)$ options. Thus, this contribution is $4(n-1)b_{2(n-1)}$. Finally, the case when the first row has no corner 1s and is not 180-degree symmetric can be made to be equivalent to a first row like $011000 \cdots 0$ by permutation invariance. The number of ways to permute the 1s is $\frac{(2n-2)(2n-4)}{2}$, avoiding the $n-i$ case, thus the contribution is $\frac{(2n-2)(2n-4)}{2}c_{2(n-1)}$. Thus, we have

$$a_{2n} = a_{2(n-1)} + (n-1)a_{2(n-2)} + 4(n-1)b_{2(n-1)} + 2(n-1)(n-2)c_{2(n-1)}.$$

Now we seek recurrences for $b$ and $c$. First consider $b_{2n}$. The outer most square onion (of dimension $(2n+2) \times (2n+2)$ because of our aforementioned convention) will have corners that look like

$$\begin{matrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & & & & & & 0 \\ 0 & & & & & & 0 \\ \vdots & & & & & & \vdots \\ 0 & & & & & & 0 \\ 0 & & & & & & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{matrix}$$

Since we seek to delete the outer-most square onion, we have deal with the 1s that look at the corner of the next inner square onion ring. Actually we can just fuse them into the next inner square onion and then delete the outer square onion. The inner square onion will now have two corner 1s. If the other corners are also occupied, then we can just delete the inner square onion, leaving a free $(2n-2) \times (2n-2)$ matrix. If the corners that we originally intended to fuse the 1s are occupied, we can use the other necessarily free corners. These two cases yield a factor 2. This contribution is then $2a_{2(n-1)}$.

Now consider when the inner square onion does not have occupied corners. There are $(2n-2)$ options for the 1 to reside. By permutation invariance, we can move it to the place next to the left corner, and then fuse the 1s facing the inner square onion corner. This situation is exactly $b_{2(n-1)}$! Thus,

$$b_{2n} = 2a_{2(n-1)} + 2(n-1)b_{2(n-1)}.$$

Finally for the even case, consider the recurrence for $c_{2n}$. First consider the case

$$\begin{matrix} 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 \end{matrix}$$

Notice the first row has form $01100\cdots0$, and the first inner square ring is like $01000\cdots0$ and the second inner square ring is all 0s. This leaves the inner most $(2n-4) \times (2n-4)$ matrix free, so this contribution is $a_{2(n-2)}$. But note that by rotating the top row of the first inner square onion by 180 degrees yields the same effect (namely $0\cdots0010$), so we have a factor 2.

$$
\begin{array}{ccccccccc}
0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & X & X & \cdots & X & X & 0 & 0 \\
0 & 0 & X & & & & X & 0 & 0 \\
\vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\
0 & 0 & X & & & & X & 0 & 0 \\
0 & 0 & X & X & \cdots & X & X & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0
\end{array}
\rightarrow \text{(fuse)}
\begin{array}{ccccccccc}
D & D & D & D & \cdots & D & D & D & D \\
D & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & D \\
D & 1 & X & X & \cdots & X & X & 0 & D \\
D & 0 & X & & & & X & 0 & D \\
\vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\
D & 0 & X & & & & X & 0 & D \\
D & 0 & X & X & \cdots & X & X & 1 & D \\
D & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & D \\
D & D & D & D & \cdots & D & D & D & D
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & X & 0 & X & \cdots & X & 0 & X & 0 \\
1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & X & 0 & & & & 0 & X & 0 \\
\vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\
0 & X & 0 & & & & 0 & X & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
0 & X & 0 & X & \cdots & X & 0 & X & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0
\end{array}
\rightarrow \text{(fuse)}
\begin{array}{ccccccccc}
1 & 1 & D & 0 & \cdots & 0 & D & 0 & 0 \\
1 & X & D & X & \cdots & X & D & X & 0 \\
D & D & D & D & \cdots & D & D & D & D \\
0 & X & D & & & & D & X & 0 \\
\vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots \\
0 & X & D & & & & D & X & 0 \\
D & D & D & D & \cdots & D & D & D & D \\
0 & X & D & X & \cdots & X & D & X & 1 \\
0 & 0 & D & 0 & \cdots & 0 & D & 1 & 1
\end{array}
$$

Now consider if the first or second inner square onion ring has a corner 1. There are 4 possibilities. Considering the case above, we see that we could fuse the remaining outer 1s onto the entry adjacent to the corner of the first inner ring and delete the outer ring, thereby reducing it to the $b_{2(n-1)}$ case. A similar procedure exists for the other cases due to permutation invariance. Explicitly, if the inner ring has an upper-left corner 1, we could delete the row and columns of the second inner ring, and then add a 1 to the upper-left corner of the outer ring to compensate for the lost 1s, yielding and equivalent scenario. In the above, $X = 0, 1$ and $D$ means deleted. However we have doubled-counted the case when inner rings both have corner 1s. In this case, all three rings can be deleted, leaving the free inner matrix with $a_{2n-4}$ possibilities. The double-counting case also has 4 possibilities. Combining with the above contribution yields a net contribution of $2a_{2n-4} + 4b_{2n-2} - 4a_{2n-4} = 4b_{2(n-1)} - 2a_{2(n-2)}$.

Finally, if the inner two rings do not satisfy the above two cases, there are two final cases. Firstly, if the 1s in the top row of the two inner rings line up, and if they don't. If they line up there are $(2n-4)$ options for the first inner ring. The second inner ring actually has two choices, $i$ and $n-i$ where $i$ is the position of the 1 in the first inner ring. This is because due to the symmetry, the bottom row of the second inner ring will have a 1 at $n-j$ where $j$ is the position of the 1 in the top row. Thus there are $2(2n-4)$ possibilities here. By symmetry, this means that not only can we delete the 3 rings, but we can also delete one more row and column due to the doubled-up ones in the two inner rings. We are then left with a $(2n-6) \times (2n-6)$ free matrix. This contribution is therefore $2(2n-4)a_{2(n-3)}$. If the other 1s in the two inner rings are *not* doubled up, then there are $(2n-4)(2n-6)$ options. Then we can delete the 3 outer rings, and fuse the 1s in the first and second inner ring into a new row, yielding a $(2k-2) \times (2k-2)$ matrix whose outer ring has two non-corner ones. The inner matrix has precisely $c_{2n-4}$

possibilities. Thus, we have

$$c_{2n} = 4b_{2(n-1)} - 2a_{2(n-2)} + 4(n-2)a_{2(n-3)} + 4(n-2)(n-3)c_{2(n-2)}.$$

Finally let's consider the odd case. Note that by symmetry, the position of one 1 in the middle row fixes the position of all 1s in the middle row and column. If the dimension is $2n+1$, the number of options for the middle row and column is then $n$. By permutation invariance, we can bring the middle 1s to the outer most ring. We can then delete the middle row and column, and fuse the middle 1s into one of the corners. If one of the corners is already occupied (2 possibilities) we choose the other corner. Then the outer ring contains only 1s, so we can delete it, leaving the inner $(2n-2)\times(2n-2)$ matrix free, so $a_{2n-2}$ possibilities. If the corner wasn't originally occupied, we can fuse the middle ones into the left corner; the other 1 has $(2n-2)$ possible options, and this case reduces to $b_{2n-2}$, But since $2a_{2(n-1)} + 2(n-1)b_{2(n-1)} = b_{2n}$, we have,

$$a_{2n+1} = n \cdot b_{2n}.$$