# Every positive integer divides a Harshad number

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A Harshad number (or Niven number) is a positive integer $n$ that is divisible by its sum of digits $s(n)$. For example, 42 is a Harshad number, because the sum of the digits of 42 is $s(42) = 4 + 2 = 6$, and 42 is divisible by 6. Harshad numbers were first investigated by D. R. Kaprekar.[2]

We will show that every positive integer $n$ divides a Harshad number. If $n$ is coprime to 10, then $10^{\phi(n)} \equiv 1 \pmod{n}$ by Euler's totient theorem.[1] Let

$$N = \frac{10^{n\phi(n)} - 1}{10^{\phi(n)} - 1} = \sum_{k=0}^{n-1} 10^{k\phi(n)} .$$

Since the digits of $N$ are 1s and 0s with exactly $n$ 1s, we see that $s(N) = n$. On the other hand, $N$ is divisible by $n$ since $10^{k\phi(n)} \equiv 1 \pmod{n}$ for all $k$. Therefore, $N$ is a Harshad number that is divisible by $n$.

Now suppose that $n$ is not coprime to 10. Then we can write $n$ uniquely as $n = mn_1$, where $m$ divides $10^k$ for some $k$, and $n_1$ is coprime to 10. By the previous argument, there exists a Harshad number $N_1$ that is divisible by $n_1$. Let $N = 10^k N_1$. Then $N$ is also a Harshad number (since $s(N) = s(N_1)$) and $N$ is divisible by $n$. This completes the proof.

## References

[1] G. Hardy, E. Wright,*An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1979.

[2] D. R. Kaprekar, *Multidigital numbers*, Scripta Math. 21, 27, 1955.