Every positive integer divides a Harshad number

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A Harshad number (or Niven number) is a positive integer \( n \) that is divisible by its sum of digits \( s(n) \). For example, 42 is a Harshad number, because the sum of the digits of 42 is \( s(42) = 4 + 2 = 6 \), and 42 is divisible by 6. Harshad numbers were first investigated by D. R. Kaprekar.[2]

We will show that every positive integer \( n \) divides a Harshad number. If \( n \) is coprime to 10, then \( 10^{\phi(n)} \equiv 1 \pmod{n} \) by Euler’s totient theorem.[1] Let

\[
N = \frac{10^{n\phi(n)} - 1}{10^{\phi(n)} - 1} = \sum_{k=0}^{n-1} 10^{k\phi(n)}.
\]

Since the digits of \( N \) are 1s and 0s with exactly \( n \) 1s, we see that \( s(N) = n \). On the other hand, \( N \) is divisible by \( n \) since \( 10^{k\phi(n)} \equiv 1 \pmod{n} \) for all \( k \). Therefore, \( N \) is a Harshad number that is divisible by \( n \).

Now suppose that \( n \) is not coprime to 10. Then we can write \( n \) uniquely as \( n = mn_1 \), where \( m \) divides \( 10^k \) for some \( k \), and \( n_1 \) is coprime to 10. By the previous argument, there exists a Harshad number \( N_1 \) that is divisible by \( n_1 \). Let \( N = 10^kN_1 \). Then \( N \) is also a Harshad number (since \( s(N) = s(N_1) \)) and \( N \) is divisible by \( n \). This completes the proof.

References
