A142978 and a family of continued fraction expansions for log(2)

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For each positive integer n we find a continued fraction expansion for the alternating sum

$$S_n = \sum_{k=1}^{\infty} \frac{(-1)^k}{p_n(k)p_n(k+1)},$$

where the polynomial $p_n(x)$ is the n-th row generating function of the square array A142978. Using a theorem of Ramanujan we show that S_n lies in $\mathbb{Q}\left(\log(2)\right)$.

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The n-th row entries of A142978 are the values $[p_n(k)]_{k\ge1}$, of the polynomial function

$$p_n(x) = \sum_{k=1}^{n} 2^{k-1}\binom{n-1}{k-1}\binom{x}{k}.$$

The first few polynomials are $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x^2$, $p_3(x) = (2x^3 + x)/3$ and $p_4(x) = (x^4 + 2x^2)/3$.

The o.g.f. for the sequence $\{p_n(x)\}$ is

$$\frac{1}{2}\frac{(1+t)^x}{(1-t)^x} - \frac{1}{2} = xt + x^2t^2 + \frac{2x^3+x}{3}t^3 + \cdots.$$

Thus the polynomial $p_n(x)$ is, apart from a constant factor, the Meixner polynomial of the first kind $\mathrm{M}_n(x; b, c)$ at $b = 0$, $c = -1$, also known as a Mittag-Leffler polynomial [3].

From the g.f. it is straightforward to show that the polynomial $p_n(x)$ is a solution of the difference equation

$$x(f(x+1) - f(x-1)) = 2nf(x)$$

normalized so that $f(1) = 1$.

Thus, for $n, k \ge 1$, we have

$$\frac{p_n(k+1) - p_n(k-1)}{p_n(k)} = \frac{2n}{k}. \qquad (1)$$

We will need the following values, easily found from the g.f.:

$$p_n(1) = 1, \quad p_n(2) = 2n. \qquad (2)$$

The following result of Euler [2, Theorem II, Section 23] represents an alternating series as a continued fraction.

Theorem. The alternating series

$$S = \frac{1}{a_1 a_2} - \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} - \frac{1}{a_4 a_5} + \cdots$$

has the continued fraction representation

$$a_1 S = \frac{1}{a_2 \,+}\ \frac{a_1}{\frac{a_3 - a_1}{a_2} \,+}\ \frac{1}{\frac{a_4 - a_2}{a_3} \,+}\ \frac{1}{\frac{a_5 - a_3}{a_4} \,+}\ \cdots .$$

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Fix a positive integer n. We set $a_k = p_n(k)$ and apply the theorem to the alternating series

$$S_n = \sum_{k=1}^{\infty} \frac{(-1)^k}{p_n(k) p_n(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{a_k a_{k+1}}.$$

Using (1) and (2), we arrive at the continued fraction representation

$$S_n = \frac{1}{2n \,+}\ \frac{1}{\frac{2n}{2} \,+}\ \frac{1}{\frac{2n}{3} \,+}\ \frac{1}{\frac{2n}{4} \,+}\ \cdots .$$

By means of an equivalence transformation this can be put in the form

$$S_n = \frac{1}{2n \,+}\ \frac{2}{2n \,+}\ \frac{6}{2n \,+}\ \frac{12}{2n \,+}\ \cdots ,$$

where the partial numerators, after the first, are of the form $m(m+1)$ for $m >= 1$.

The value of S_n follows from a result of Ramanujan [1, Chapter 12, Entry 32(i)] :

$$S_n = 1 + (-1)^{n+1} 2n \left(\log(2) - 1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n} \right).$$

The first few cases are

$$S_1 = \frac{1}{2 +}\frac{2}{2 +}\frac{6}{2 +}\frac{12}{2 +}\ \cdots\ = 2\log(2) - 1$$

$$S_2 = \frac{1}{4 +}\frac{2}{4 +}\frac{6}{4 +}\frac{12}{4 +}\ \cdots\ = 3 - 4\log(2)$$

$$S_3 = \frac{1}{6 +}\frac{2}{6 +}\frac{6}{6 +}\frac{12}{6 +}\ \cdots\ = 6\log(2) - 4$$

$$S_4 = \frac{1}{8 +}\frac{2}{8 +}\frac{6}{8 +}\frac{12}{8 +}\ \cdots\ = \frac{17}{3} - 8\log(2)$$

$$S_5 = \frac{1}{10 +}\frac{2}{10 +}\frac{6}{10 +}\frac{12}{10 +}\ \cdots\ = 10\log(2) - \frac{41}{6}$$

References

[1] Bruce C. Berndt, Ramanujan's Notebooks Part II, Springer-Verlag.

[2] L. Euler, On the Transformation of Infinite Series to Continued Fractions. Translated by Daniel W. File.

[3] Eric W. Weisstein, Meixner polynomial of the first kind