Making any integer with four 2s (https://eli.thegreenplace.net/2025/making-any-integer-with-four-2s/)

February 22, 2025 at 14:53 **Tags** Math (https://eli.thegreenplace.net/tag/math)

There's a cute math puzzle that can be interesting to folks on very different levels:

> Given exactly four instances of the digit 2 and some target natural number, use any mathematical operations to generate the target number with these 2s, using no other digits.

Some examples can be done by elementary school kids:

$$1 = \frac{2+2}{2+2}$$

$$2 = \frac{2}{2} + \frac{2}{2}$$

$$3 = 2 \cdot 2 - \frac{2}{2}$$

$$4 = 2 + 2 + 2 - 2$$

$$5 = 2 \cdot 2 + \frac{2}{2}$$

$$6 = 2 \cdot 2 \cdot 2 - 2$$

In middle school, kids learn about exponents, factorials, etc. which expands the range considerably:

$$18 = 2^{2^2} + 2$$

$$28 = (2+2)! + 2 + 2$$

$$256 = (2+2)^{2+2}$$

$$65536 = 2^{2^{2^2}}$$

Then come the tricks; for example, the number 22 (twenty two) can be seen as a valid use of two 2s, and so on; so we can have:

$$26 = 22 + 2 + 2$$
$$11 = \frac{22}{\sqrt{2+2}}$$
$$444 = 222 \cdot 2$$

Getting to 7 is notoriously difficult, but if you allow even more mathematical tools like the Gamma function (https://en.wikipedia.org/wiki/Gamma_function), it becomes easy:

$$7 = \Gamma(2) + 2 + 2 + 2$$

The more math skill people have, the more numbers they can make. See this thread (https://math.stackexchange.com/questions/1034122/get-the-numbers-from-0-30-by-using-the-number-2-four-times) for some fun concoctions using integrals, repeating fractions and combinatorial operators. One of my favorite examples involves complex numbers:

$$12 = |2 + 2\sqrt{-2}|^2$$

So the fun doesn't end even after one graduates from university! In fact, this seems to have been a favorite pastime for mathematicians in the 1920s. Until Paul Dirac (https://en.wikipedia.org/wiki/Paul_Dirac) ruined it for everyone by finding a general solution for every number.

It's all about nested square roots:

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{2^{-1}}$$
$$\sqrt{\sqrt{2}} = 2^{\frac{1}{4}} = 2^{2^{-2}}$$
$$\sqrt{\sqrt{\sqrt{2}}} = 2^{\frac{1}{8}} = 2^{2^{-3}}$$

If the square root is applied *n* times:

$$\sqrt{\sqrt{\cdots n \cdots \sqrt{2}}} = 2^{2^{(-n)}}$$

All that's left now is some base-2 logarithms:

$$log_2 2^{2^{(-n)}} = 2^{(-n)}$$

And another:

$$log_2(log_2 2^{2^{(-n)}}) = -n$$

This leads to the general formula:

$$n = -log_2\left(log_2\left(\sqrt{\sqrt{\cdots n \cdots \sqrt{2}}}\right)\right)$$

There's just one small wrinkle: it uses *three* instances of the digit 2, not four. This is easy to amend, however; since $2 = \sqrt{2+2}$, we can replace any single digit with that and get exactly four:

$$n = -log_{\sqrt{2+2}}\left(log_2\left(\sqrt{\sqrt{\cdots n \cdots \sqrt{2}}}\right)\right)$$

One may claim this is cheating, but it seems to be in line with the rules of the puzzle! Note that the entity *n* doesn't actually appear anywhere - it's just a helper to count the number of repeated square roots. For example, another way to express 7 is:

$$7 = -log_{\sqrt{2+2}}\left(log_2\left(\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}}}\right)\right)$$

There are exactly four 2s, and this uses only reasonable, elemental math operations to do the calculation. It's clear that *any* number can be expressed this way; the only challenge is properly drawing all those square roots!

Credits

I've read about this story in Graham Farmelo's book *The Strangest Man: The Hidden Life of Paul Dirac, Quantum Genius*. I'm enjoying this book so far.

For comments, please send me an email (mailto:eliben@gmail.com).

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