$A135061(n) = \lfloor 2n^{3/2} \rfloor$

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Theorem 1 *For any positive integer n, the minimum value of $\lfloor n^3/m \rfloor + m$ for positive integers m is $\lfloor 2n^{3/2} \rfloor$, and this occurs at $m = \lceil n^{3/2} \rceil$.*

Proof Let $f(x) = n^3/x + x$. This is convex on $(0, \infty)$ with minimum value $2n^{3/2}$ at $x = n^{3/2}$. Thus for positive integers m, $f(m) \ge 2n^{3/2}$ so that $\lfloor n^3/m \rfloor + m \ge \lfloor 2n^{3/2} \rfloor$. The claim is that equality occurs for $m = \lceil n^{3/2} \rceil$.

We have $0 \le m - n^{3/2} < 1$. Using Taylor's theorem and the fact that $f''(x) = 2n^3/x^3$ is decreasing,

$$f(m) \le f(n^{3/2}) + \frac{f''(n^{3/2})}{2}(m - n^{3/2})^2 \le 2n^{3/2} + n^{-3/2}$$

In order for $\lfloor n^3/m \rfloor + m > \lfloor 2n^{3/2} \rfloor$, we would need there to be some integer k with

$$2n^{3/2} < k \le 2n^{3/2} + n^{-3/2}$$

and thus

$$4n^3 < k^2 \le \left(2n^{3/2} + n^{-3/2}\right)^2 = 4n^3 + 4 + n^{-3}$$

Since $0 < n^{-3} < 1$, this says n and k satisfy one of the Diophantine equations

$$k^2 = 4n^3 + j$$

where $j = 1, 2, 3$ or 4. Now $k^2 \equiv 0$ or $1 \bmod 4$, so only $j = 1$ and $j = 4$ are possible.

If k and n are integers satisfying the equation

$$k^2 = 4n^3 + 1$$

then $s = 4n$ and $t = 4k$ are integers satisfying the Mordell equation

$$t^2 = s^3 + 16$$

Since $A081119(16) = 2$, there are only two integer solutions to that equation: these are easily found to be $t = \pm 4$, $s = 0$, corresponding to $n = 0$. So there are no examples of this case with $n \geq 1$.

Similarly, from $k^2 = 4n^3 + 4$ with $s = 4n$ and $t = 4k$ we get the Mordell equation

$$t^2 = s^3 + 64$$

with $A081119(64) = 5$. Here the five integer solutions are $s = -4, t = 0$, $s = 0, t = \pm 8$ and $s = 8, t = \pm 24$, corresponding to $n = -1, 0, 2$ respectively. But for $n = 2$, $\lfloor 8/3 \rfloor + 3 = \lfloor 2 \cdot 2^{3/2} \rfloor = 5$, so this is not a counterexample.

This completes the proof.