Proper Self-Containing Sequences, Fractal Sequences, and Para-Sequences

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August 20, 2007

Abstract

Various kinds of ordered sets are discussed: proper self-containing sequences, infinitive sequences, fractal sequences, and para-sequences (which include all sequences as well as ordered sets that are not sequences). The following are defined: position array, normalized fractal sequence, placement sequence, dense para-sequence, and dense fractal sequence. Relationships among these are presented, and among the 23 examples, the Cantor fractal sequence and the Farey fractal sequence are introduced.

1 Introduction

A sequence that contains itself as a proper subsequence obviously does so infintely many times. This sort of containment-nest is analogous to nested geometric configurations widely known as fractals. In this article, we examine first the class of proper self-containing sequences, and then, as a subclass, fractal sequences (e.g., [1],[3],[4],[6]). Then we introduce a generalization of sequence called a *para-sequence* and show relationships between para-sequences and fractal sequences.

In Sections 2 and 3, proper self-containing sequences, in which the terms may be arbitrary objects, are shown to be essentially in one-to-one correspondence with certain position arrays consisting of the positive integers, each occurring exactly once. In Section 4, regular self-containing sequences and their duals are defined. Section 5 discusses fractal sequences that naturally arise in connection with the Cantor ternary set and the set of Farey fractions.

In Section 6, normalized fractal sequences are introduced, and their description in terms of permutations of sets of the form $\{1, 2, 3, \ldots, n\}$ is given. Para-sequences are defined in Section 7 in terms of an order relation $\prec$, and associated placement sequences are discussed in Section 8. Section 9 introduces dense para-sequences and dense fractal sequences. Also in Section 9, the signature sequence of an arbitrary positive irrational number θ is a dense fractal sequence. Similar results are given in Example 22 for fractional parts $\{h\theta\}$. In Section 10 the cut-point of a para-sequence is defined, leading to pairs of complementary sequences.

2 Proper Self-Containing Sequences

A *proper self-containing sequence* (PSCS) is a sequence (a_n) that contains a proper subsequence (a_{n_i}) that is identical (a_n):

$$a_{n_i} = a_i \text{ for all } i \text{ in the set } N \text{ of natural numbers: } 1, 2, 3, \ldots.$$

Clearly, a PSCS properly contains itself infinitely many times:

$$a_{n_i} = a_i, \quad a_{n_{i_j}} = a_{i_j}, \ldots \;.$$

The terms may be rational numbers or pebbles or indentations along a coastline, but unless otherwise implied, we shall assume that the terms of each PSCS belong to N. In that case, recall [4] that a sequence $x = (x_n)$ in N is an *infinitive sequence* if for every i,

(F1) $x_n = i$ for infinitely many n. Let $T(i, j)$ be the jth index n for which $x_n = i$.

It is easy to prove that a sequence containing every positive integer is a PSCS if and only the sequence is an infinitive sequence. It is also easy to prove that if (a_{n_i}) is a proper subsequence of a PSCS (a_n) as in the definition of PSCS, then the complement of (a_{n_i}), meaning the sequence that remains after the terms a_{n_i} are removed from (a_{n_i}), is also a PSCS.

Example 1. Suppose (a_n) is periodic. Then (a_n) is a PSCS. In particular, $(1, 1, 1, \ldots)$ is a PSCS.

Example 2. Let a_n be the number of 0s and 1s occurring from right to left to reach the first 1 in the base-2 representation of n :

$$(a_n) = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, \ldots).$$

This sequence, A001511 in the Online Encyclopedia of Integer Sequences [7], is a PSCS.

3 Position Arrays

We shall show that PSCSs are essentially in one-to-one correspondence with certain arrays whose terms are in N, in the sense that such an array begets a family of essentially equivalent PSCSs, and every PSCS begets such an array.

For $m \in N$, let $N_m = \{1, 2, 3, \ldots, m\}$ and $N_\infty = N$. Suppose, for $(i, j) \in N_m \times N$, where $1 \leq m \leq \infty$, that $T = (T(i, j))$ is an array that partitions N into a sequence of increasing sequences; i.e.,

(P1) if $n \in N$, then $n = T(i, j)$ for some (i, j);
(P2) for each i, the sequence $(T(i, j))$ is increasing;
(P3) if $i_1 < i_2$, then no term of $(T(i_1, j))$ is a term of $(T(i_2, j))$.

Next, suppose $\{\mathcal{O}_i\}$, for $i \in N_m$, is a set of objects, and define

$$a_{T(i,1)} = \mathcal{O}_i.$$

If $m = 1$, then $a_n = \mathcal{O}_1$ for all $n \in N$, and (a_n) is a PSCS, as in Example 1. Assume, then, that $m \geq 2$, and define

$$a_{T(i,j)} = a_{T(i,1)} \tag{1}$$

for all $j \in N$. By (P1), a_n is defined for every $n \in N$. For any such n, let i be the number such that $n = T(i,j)$ for some j. Let $n_j = T(i, j+1)$, so that $a_{n_j} = a_n$. Then (a_{n_j}) is a proper subsequence of (a_n) such that (a_{n_j}) and (a_n) are identical. Call T the *position array of* (a_n).

We have just seen how a position array begets a PSCS — indeed, a family of PSCSs, depending on $\{\mathcal{O}_i\}$. Conversely, a PSCS (a_n) yields a position array, as follows: call h and i equivalent if $a_h = a_i$, and partition N into equivalence classes; arrange each class in increasing order, thus obtaining a (possibly finite) set of increasing sequences. To form T, let

(row 1) = the sequence having first term a_1;
(row 2) = the sequence whose first term is the first of (a_n) that is not in row 1;

and for $i \geq 2$, let (row i) be the sequence whose first term is the first in (a_n) that is not in rows 1 to $i-1$. This ordering of rows yields an array T, for which properties (P1)-(P3) clearly hold.

Example 3. The position array T for the sequence (a_n) in Example 2 is given by

$$T = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & \cdots \\ 2 & 6 & 10 & 14 & 18 & 22 & 26 & 30 & \cdots \\ 4 & 12 & 20 & 28 & 36 & 44 & 52 & 60 & \cdots \\ 8 & 24 & 40 & 56 & 72 & 88 & 104 & 120 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

the transpose of the array A054582.

4 Regular Sequences and Arrays

A position array is a *regular position array* if it has infinitely many rows and every column is increasing. A proper self-containing sequence is a *regular self-containing sequence* (RSCS) if its position array is regular. For example, the sequence (a_n) in Example 2 is an RSCS.

Clearly if T is regular, then its transpose, T^*, is regular. If (a_n) is an RSCS with position array T, then T^* is the position array of an RSCS, which we denote by (a_n^*) and call the *dual* of (a_n). As $(T^*)^* = T$, we have $((a_n^*)^*) = (a_n^*)$.

Example 4. Let $(a_n) =$ A001511, as in Examples 2 and 3. Then

$$\begin{aligned}(a_n^*) &= (1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 4, 8, 1, 9, 5, 10, 3, \ldots) \\ &= \text{A003602}.\end{aligned}$$

The position array of (a_n^*), namely the transpose of the array T in Example 3, is the dispersion [2] of the even positive integers, given by A054582.

Example 5. If $(a_n) =$ A002260, then $(a_n^*) =$ A004736.

Example 6. Recall [4] the definition of a signature sequence:

> For any irrational number $\theta > 0$, let
>
> $$S(\theta) = \{c + d\theta : \quad c \in N, \ d \in N\},$$
>
> and let $c_n(\theta) + d_n(\theta)\theta$ be the sequence obtained by arranging the elements of $S(\theta)$ in increasing order. A sequence a is a *signature* if there exists a positive irrational number θ such that $a = (c_n(\theta))$. In this case, a is the *signature of* θ.

The dual of a signature is also a signature; specifically, if (a_n) is the signature of θ, then (a_n^*) is the signature of θ^{-1}. For example, the signature of $2^{1/2}$ is

$$(1, 2, 1, 3, 2, 1, 4, 3, 2, 5, 1, 4, 3, 6, 2, 5, 1, 4, 7, 3, 6, 2, 5, 8, 1, \ldots) = \text{A007336},$$

and that of $2^{-1/2}$ is

$$(1, 1, 2, 1, 2, 3, 1, 2, 3, 1, 4, 2, 3, 1, 4, 2, 5, 3, 1, 4, 2, 5, 3, 1, \ldots) = \text{A023115}.$$

5 Fractal Sequences

A search of [7] for "fractal sequence" reveals that in recent years, various different (kinds of) sequences have been called "fractal" and that what many of them have in common is that they are PSCSs, or equivalently, infinitive sequences. In this article, a fractal sequence is a special kind of RSCS defined ([3],[4] [8]) as an infinitive sequence x as in (F1) such that the following two properties also hold:

(F2) if $i + 1 = x_n$, then there exists $m < n$ such that $i = x_m$;
(F3) if $h < i$, then for every j, there is exactly one k such that

$$T(i, j) < T(h, k) < T(i, j + 1).$$

According to (F2), the first occurrence of each $i > 1$ in x must be preceded at least once by each of the numbers $1, 2, ..., i - 1$, and according to (F3), between consecutive occurrences of i in x, each h less than i occurs exactly once. Of course, the array here denoted by T is the placement array of (x_n).

Examples of fractal sequences are (a_n^*) in Example 4 but not (a_n), both (a_n) and (a_n^*) in Example 5, and all signatures, as in Example 6. Other examples, of quite a variety of kinds, are given by these sequences in [7]: A003603, A022446, A022447, A023133, A108712, A120873, A120874, A122196, A125158.

Example 7. In the unit interval $[0, 1]$, the classical Cantor set is the set of rational numbers whose base-3 representation consists solely of 0's and 2's.

This Cantor set sometimes called the prototype of a (geometric) fractal. We shall arrange the numbers of the Cantor set to form a PSCS. First, write the numbers, in base 3, as follows:

$$\begin{aligned}
&0,\\
&0, .2,\\
&0, .02, .2,\\
&0, .02, .2, .22,\\
&0, .002, .02, .2, .22,\\
&0, .002, .02, .2, .202, .22, \text{ and so on,}
\end{aligned}$$

and then concatenate these segments to form the PSCS

$$c = (0, 0, .2, 0, .02, .2, 0, .02, .2, .22, 0, .002, .02, .2, .22, 0, .002, .02, .2, .202, .22, \ldots).$$

As suggested by the definition of PSCS, if the terms c_n of a PSCS are in one-to-one correspondence with objects a_n, then we obtain essentially the same PSCS by replacing each c_n by a_n. For the sequence, $c = (c_n)$ let a_n be the position in c at which c_n first appears, where each distinct predecessor is counted only once, yielding the Cantor fractal sequence and its associated interspersion:

$$(a_n) = (1, 1, 2, 1, 3, 2, 1, 3, 2, 4, 1, 5, 3, 2, 4, 1, 5, 3, 2, 6, 4, \ldots) = \text{ A088370}$$

and

$$T = \begin{pmatrix}
1 & 2 & 4 & 7 & 11 & 16 & 22 & \cdots\\
3 & 6 & 9 & 14 & 19 & 26 & 33 & \cdots\\
5 & 8 & 13 & 18 & 24 & 31 & 40 & \cdots\\
10 & 15 & 21 & 28 & 35 & 44 & 55 & \cdots\\
12 & 17 & 23 & 30 & 39 & 48 & 58 & \cdots\\
20 & 27 & 34 & 43 & 53 & 64 & 75 & \cdots\\
25 & 32 & 41 & 50 & 61 & 72 & 85 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = \text{A131966.}$$

Example 8. Here we introduce a Farey fractal sequence, much in the spirit of Example 7, as this sequence consists of positions of entries among the Farey fractions, which are represented by the following table:

Farey fractions of order 1: $\frac{0}{1}$ $\frac{1}{1}$

Farey fractions of order 2: $\frac{0}{1}$ $\frac{1}{2}$ $\frac{1}{1}$

Farey fractions of order 3: $\frac{0}{1}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{1}{1}$

Farey fractions of order 4: $\frac{0}{1}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{1}{1}$

Farey fractions of order 5: $\frac{0}{1}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{2}{5}$ $\frac{1}{2}$ $\frac{3}{5}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$ $\frac{1}{1}$

Concatenating gives the PSCS

$$(\frac{0}{1}, \frac{1}{1}, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \ldots).$$

Replace each fraction in this sequence by the position at which it first appears, where each distinct predecessor is counted only once, getting

$$\begin{pmatrix} 1 & 2 & & & & & & & & & \\ 1 & 3 & 2 & & & & & & & & \\ 1 & 4 & 3 & 5 & 2 & & & & & & \\ 1 & 6 & 4 & 3 & 5 & 7 & 2 & & & & \\ 1 & 8 & 6 & 4 & 9 & 3 & 19 & 5 & 7 & 11 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad (2)$$

Concatenating gives the Farey fractal sequence:

$$(1, 2, 1, 3, 2, 1, 4, 3, 5, 2, 1, 6, 4, 3, 5, 7, 2, 1, 8, 6, 4, 9, 3, 10, 5, 7, 11, 2, 1, 12,$$
$$8, 6, 4, 9, 3, 10, 5, 7, 11, 13, 2, 1, 14, 12, 8, 6, 15, 4, 9, 16, 3, 17, \ldots) = \text{ A131967},$$

with position array (i.e., the associated interspersion)

$$\begin{pmatrix} 1 & 3 & 6 & 11 & 18 & \cdots \\ 2 & 5 & 10 & 17 & 28 & \cdots \\ 4 & 8 & 14 & 23 & 35 & \cdots \\ 7 & 13 & 21 & 33 & 48 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \text{ A131968.}$$

6 Normalization and Placement Sequences

Every fractal sequence $a = (a_n)$ can be separated into segments which each begin with 1 and which contain 1 only at the beginning, as in (2). Given such a segmentation, three things can happen which we wish to modify.

First, if $a_2 \neq 1$, then place an initial 1 before a_1 (which is necessarily 1), so that the new sequence begins with $1, 1$. Second, a segment may be repeated, as in A023115 in Example 6, where the segment $(1, 2, 3)$ occurs twice. In such a case, retain only the first of repeated segments in each case where repetition occurs. To keep things simple, we abuse notation by letting (a_n) denote the sequence (still a fractal sequence) which remains after the first two steps have been applied to the original sequence a.

Third, we shall arrange for the mth segment to be of length m. To begin, the 1st segment already consists of 1 by itself. The second segment is necessarily a permutation $(a_2, a_3, \ldots, a_m)$ of $(1, 2, \ldots, m)$ for some $m \geq 2$, where $a_2 = 1$. Replace this segment with the following segments, consecutively if $m \geq 4$:

$$(a_2, a_3), \quad (a_2, a_3, a_4), \quad (a_2, a_3, a_4, a_5), \quad \ldots, \quad (a_2, a_3, a_4, a_5, \ldots, a_m).$$

Repeat the procedure on the next segment, $(a_{m+1}, a_{m+2}, \ldots, a_{m+q})$, where $a_{m+1} = 1$, and continue in this manner inductively. The resulting sequence is the *normalized fractal sequence of* a, denoted by $\mathcal{N}(a)$. Although possibly $\mathcal{N}(a) \neq a$, clearly infinitely many of the segments of $\mathcal{N}(a)$ are also segments of a, so that in this sense, a is equivalent to $\mathcal{N}(a)$.

For all $m \in N$, the m^{th} segment of $\mathcal{N}(a)$ is a permutation of $(1, 2, \ldots, m)$, so that the $(m+1)^{st}$ segment arises from the m^{th} according to the placement of $m+1$ among the available $m+1$ places. Let $\mathcal{P}(a)$, called the *placement sequence of* a (and of $\mathcal{N}(a)$ and any other fractal sequence b such that $\mathcal{N}(b) = \mathcal{N}(a)$), denote the sequence whose m^{th} term is the position of m in the m^{th} segment of $\mathcal{N}(a)$.

Example 9. Let a be the signature of $2^{1/2}$, as in Example 6:

$$\begin{aligned} a &= (1,2,1,3,2,1,4,3,2,5,1,4,3,6,2,5,1,4,7,3,6,2,5,8,1,\ldots) \\ \mathcal{N}(a) &= (1,1,2,1,3,2,1,4,3,2,1,4,3,2,5,1,4,3,6,2,5,1,4,7,3,6,2,5,\ldots) \\ \mathcal{P}(a) &= (1,2,2,2,5,4,3,8,\ldots). \end{aligned}$$

Example 10. Let a be the Cantor fractal sequence, as in Example 7. In this case, $\mathcal{N}(a) = a$, and

$$\mathcal{P}(a) = (1,2,2,4,2,5,4,4,8,2,7,4,11,4,\ldots).$$

Example 11. Let a be the Farey fractal sequence, as in Example 8. Then

$$\begin{aligned} \mathcal{N}(a) &= (1,1,2,1,3,2,1,4,3,2,1,4,3,5,2,1,6,4,3,5,2,1,1,6,4,3,5,7,2,\ldots) \\ \mathcal{P}(a) &= (1,2,2,2,4,2,6,2,5,7,10,2,\ldots). \end{aligned}$$

Regarding Examples 10 and 11, in each case the transpose of the placement array is not regular, so that (a_n^*), although a PSCS, is not a fractal sequence, and $\mathcal{N}(a^*)$ and $\mathcal{P}(a^*)$ are undefined. Recalling that a placement array T belongs to a fractal sequence if and only if T is a dispersion (or equivalently, an interspersion), it is natural to seek a characterization of dispersions T for which T^* is also a dispersion, so that, for example, (a_n^*) is a fractal sequence. Such a characterization is given in [5].

Example 12. Using the sequence

$$\mathcal{P} = (1,2,2,3,3,4,4,5,5,6,6,7,7,8,8,\ldots)$$

as a placement sequence yields the normalized fractal sequence

$$a = (1,1,2,1,3,2,1,3,4,2,1,3,5,4,2,1,3,5,6,4,2,1,3,5,7,6,4,2,1,\ldots),$$

which is related to the Smarandache Permutation Sequence, A004741, in a close and obvious manner.

Example 13. Using the sequence

$$\mathcal{P} = (1,2,2,4,3,6,4,8,5,10,6,12,7,14,8,16,9,\ldots)$$

as a placement sequence yields the normalized fractal sequence

$$a = (1, 1, 2, 1, 3, 2, 1, 3, 2, 4, 1, 3, 5, 2, 4, 1, 3, 5, 2, 4, 6, 1, 3, 5, 7, 2, 4, 6, 1, \ldots),$$

which is closely related by A087467.

Examples 12 and 13 illustrate the fact that if $\mathcal{P} = (p_n)$ is a sequence satisfying $p_1 = 1$ and $2 \leq p_n \leq m$ for all $m \geq 2$, then $\mathcal{P}$ is the placement sequence of a normalized fractal sequence.

Referring to (F1) in Section 3, the array T is called ([3, 4]) an interspersion. Clearly, if T is the interspersion of a normalized fractal sequence then each diagonal of T is a permutation of consecutive integers. Specifically, diagonal n is a permutation of the integers

$$\binom{n}{2} + 1, \binom{n}{2} + 2, \ldots, \binom{n+1}{2}.$$

7 Para-sequences

Suppose S is a set and $f : S \times S \to \{0, 1\}$ satisfies

$$f(i, j) + f(j, i) = 1$$

for all i and j such that $i \neq j$. In case $S \subseteq N$, the function f is a *para-sequence*. Thus, a para-sequence is essentially an order relation, but in this section, we shall be primarily interested in a form of representation of f as a sequence (which may or may not be esssentially f itself). A para-sequence f is determined by and determines a relation $\prec$: we write $i \prec j$ to mean $f(i, j) = 0$ and $i \neq j$, and we and refer to $\prec$ as the para-sequence.

First, we shall show that every sequence in N is essentially a para-sequence. Suppose that (a_n) is a sequence in N, let $S = \{a_n\}$; that is, the set of numbers that are terms of (a_n). Define f by

$$f(a_i, a_j) = \begin{cases} 0 & \text{if } i \leq j \\ 1 & \text{if } i > j. \end{cases}$$

Introduce a relation $\prec$ in S, called *precedes* or *is to the left of* by

$$a_i \prec a_j \quad \text{if} \quad f(a_i, a_j) = 0 \text{ and } i \neq j.$$

Clearly

$$a_1 \prec a_2 \prec a_3 \prec \cdots,$$

so that $\prec$, and f, are essentially the sequence (a_n).

As a first example of a para-sequence that is essentially not a sequence, start with a sequence (a_n) and define $\prec$ on the set $\{a_n\}$ by

$$a_i \prec a_j \text{ if } i > j;$$

here, $\prec$ is essentially the reversal of (a_n). It is mentioned above that a para-sequence can be represented as a certain sequence. Consider, as an example, the sequence $(a_n) = (1, 2, 3, \ldots)$, for which $\prec$ can be represented as the non-sequence $(\ldots, 3, 2, 1)$. Break the latter into a segments, $1, 21, 321, 4321, \ldots$ and concatenate:

$$(1, 2, 1, 3, 2, 1, 4, 3, 2, 1, \ldots) = \text{ A004736},$$

this being a sequence in which $\prec$ is represented as a "growing block"

$$(n, n-1, \ldots, 3, 2, 1);$$

one may think of $\prec$ as a sort of limit of the growing block.

Example 14. Consider this sequence of finite sequences:

$$1,\ 213,\ 42135,\ 6421357, \ldots, \tag{3}$$

where, if s_n is the nth of these, then s_{n+1} is obtained by placing $2n-2$ before s_n and $2n-1$ after s_n. In (3), "213", for example, represents the ordered triple $(2, 1, 3)$. Loosely speaking, the anticipated para-sequence is a "limit" of (3), but a more precise and generally applicable version follows from the fact that the precedence relation $\prec$ is preserved as we pass from each s_n to s_{n+1}. For the present example, the para-sequence is given by

$$f(i,j) = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i \text{ is even and } j \text{ odd} \\ 0 & \text{if } i, j \text{ are even and } i > j \\ 0 & \text{if } i, j \text{ are odd and } i < j \\ 1 & \text{otherwise.} \end{cases}$$

The concatenation of the finite sequences in (3) is the Smarandache Permutation Sequence, A004741, alternatively defined in [7] as the concatenation of the sequences

$$(1, 3, \ldots, 2n-1, 2n, 2n-2, \ldots, 2).$$

8 Placement sequences

A para-sequence (and its precedence relation $\prec$) begets a *P-placement sequence* p as follows:

$$p(1) = 1$$

$$p(2) = \begin{cases} 1 & \text{if } 2 \prec 1 \\ 2 & \text{if } 1 \prec 2 \end{cases}$$

$$p(3) = \begin{cases} 1 & \text{if } 3 \prec 2 \prec 1 \text{ or } 3 \prec 1 \prec 2 \\ 2 & \text{if } 2 \prec 3 \prec 1 \text{ or } 1 \prec 3 \prec 2 \\ 3 & \text{if } 1 \prec 2 \prec 3 \text{ or } 2 \prec 1 \prec 3 \end{cases}$$

$$\vdots$$

That is, $p(n)$ is the position of n when the numbers $1, 2, 3, \ldots, n$ are arranged from left to right using the "to the left of" relation $\prec$. Note that the identity sequence, given by $p(n) = n$, corresponds to $\prec=<$ and that in this case, the para-sequence is essentially $(1, 2, 3, \ldots)$.

To summarize, a para-sequence (that is, $\prec$) determines a P-placement sequence. Clearly the converse holds: every sequence p satisfying

$$1 \leq p(m) \leq n, \text{ for } m = 1, 2, \ldots, n,$$

determines a para-sequence, so that we have a one-to-one correspondence between para-sequences and P-placement sequences.

Given a P-placement sequence p, it will be helpful to define the *block-sequence*, $(s(n))$ of p by $s(1) = 1$, and for $n \geq 2$, the block $s(n)$ is formed from $s(n-1)$ by placing n in position $p(n)$; e.g., if $s(3) = 213$ and $p(4) = 1$, then $s(4) = 4213$. Thus, the P-placement sequence for f in Example 14 is

$$(1, 1, 3, 1, 5, 1, 7, 1, 9, 1, \ldots)$$

and the block sequence is

$$(1,\ 21,\ 213,\ 4213,\ 42135, \ldots).$$

The concatenation $s(1)s(2)s(3)\ldots$ (with commas inserted) is the *fractal sequence associated with* p, denoted by F_p. Separating the terms of F_p into segments beginning with 1, we obtain the *segment-sequence associated with* p *(and with* F_p*)*.

Example 15. As a P-placement sequence,

$$p(n) = (1, 2, 1, 4, 1, 6, 1, 8, \ldots)$$

determines the para-sequence

$$f(i,j) = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i \text{ is even and } j \text{ is odd} \\ 0 & \text{if } i, j \text{ are both odd and } i > j \\ 0 & \text{if } i, j \text{ are both even and } i < j \\ 1 & \text{otherwise,} \end{cases}$$

with block-sequence

$$1,\ 12,\ 312,\ 3124,\ 53124,\ 531246, \ldots), \tag{4}$$

fractal sequence

$$(1, 1, 2, 3, 1, 2, 3, 1, 2, 4, 5, 3, 1, 2, 4, 5, 3, 1, 2, 4, 6, \ldots), \tag{5}$$

and segment-sequence

$$(1, 123, 123, 12453, 12453, \ldots). \tag{6}$$

Example 16. Let

$$q = (12, 3124, 531246, 75312468, \ldots), \tag{7}$$

where each term s_n begets s_{n+1} be prefixing $2n-1$ and suffixing $2n$ to s_n. The concatenation is the fractal sequence

$$1, 2, 3, 1, 2, 4, 5, 3, 1, 2, 4, 6, 7, 5, 3, 1, 2, 4, 6, \ldots), \tag{8}$$

much like that in Example 15; indeed (7) consists of the even-numbered terms of (4). The main point here is that the relation $\prec$ corresponding to p in Example 15 is identical to that corresponding to q in Example 16. The representation $\prec$, or f, is a better way to represent the para-sequence than a fractal sequence such as (5) or (8), because for any given $\prec$, there are many such sequences. On the other hand, a fractal sequence such as (5) or (8) is helpful when one wishes to think of $\prec$ (and f) as a sort of limit.

Note that the segment-sequence (6) contains repeated blocks. Expelling duplicates leaves

$$(1, 123, 12453, 1246753, 124689753, \ldots),$$

which we modify so that the nth term is a permutation of $12\ldots n$:

$$(1, 12, 123, 1243, 12453, 124653, 1246753, \ldots), \tag{9}$$

this being, after insertion of commas, a normalized fractal sequence, from which we read an F-placement sequence:

$$(1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, \ldots).$$

Example 17.

P-placement sequence	$(1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 8, \ldots)$
block-sequence	$(1, 12, 312, 3142, 53142, 531642, 7531642, \ldots)$
segment-sequence	$(1, 123, 123, 14253, 14253, 1642753, 1642753, \ldots)$
fractal sequence	$(1, 1, 2, 3, 1, 2, 3, 1, 4, 2, 5, 3, 1, 4, 2, 5, 3, 1, 6, 4, \ldots)$
normalized fractal sequence	$(1, 1, 2, 3, 1, 4, 2, 5, 3, 1, 6, 4, 2, 7, 5, 3, \ldots)$
normalized segment-sequence	$(1, 12, 123, 1423, 14253, 164253, 1642753, \ldots)$
F-placement sequence	$(1, 2, 3, 2, 4, 2, 5, 2, 6, 2, 7, 2, 8, 2, 9, 2, 10, 2, \ldots)$
para-sequence	$f(i,j) = \begin{cases} 0 & \text{if } (i \text{ is odd and } j \text{ is even}) \text{ or } i = j \\ 0 & \text{if } i \text{ and } j \text{ are both odd and } i > j \\ 0 & \text{if } i \text{ and } j \text{ are both even and } i > j \\ 1 & \text{otherwise} \end{cases}$

Example 18.

P-placement sequence	$(1,1,2,2,3,3,4,4,5,5,6,6,7,7,8,8,9,9,\ldots)$
block-sequence	$(1,21,231,2431,24531,246531,2467531,\ldots)$
segment-sequence	$(12,123,1243,12453,124653,1246753,\ldots)$
fractal sequence	$(1,1,2,1,2,3,1,2,4,3,1,2,4,5,3,1,2,4,6,5,3,1,2,\ldots)$
F-placement sequence	$(1,2,3,3,4,4,5,5,6,6,7,7,8,8,9,9,10,10,\ldots)$
para-sequence	$f(i,j)=\begin{cases} 0 & \text{if } i=1 \text{ or } i=j \\ 0 & \text{if } i \text{ and } j \text{ are both even and } i<j \\ 0 & \text{if } i \text{ and } j \text{ are both odd and } i>j \\ 1 & \text{otherwise} \end{cases}$

9 Dense para-sequences

A para-sequence f is *dense* if for the following three conditions hold:

(D1) for every i, there exists k such that $k \prec i$;
(D2) for every i and j satisfying $i \prec j$, there exists k such that $i \prec k \prec j$;
(D3) for every j, there exists k such that $j \prec k$.

Similarly, a fractal sequence is *dense* if (D2) and (D3) hold.

Example 19. To construct a dense para-sequence, start with 1, and then isolate it using $2,3$ like this: $2,1,3,$. Then isolate these using $4,5,6,7$ like this:

$$4,2,5,1,6,3,7$$

and so on. Concatenate to obtain the dense para-sequence represented by the sequence

$$(1,2,1,3,4,2,5,1,6,3,7,8,4,9,2,10,5,11,1,12,6,13,3,14,7,15,\ldots) = \text{A131987}.$$

Example 20. Suppose $(c_n(\theta))$ is the signature of an irrational number $\theta > 0$, as in Example 6. We shall show that this fractal sequence is dense. Suppose that $i \prec j$; i.e., that

$$i + i_1\theta < j + j_1\theta < i + (i_1+1)\theta$$

for some $i_1 \in N$ and $j_1 \in N$. We wish to prove that there exist h, k, k_1 in N such that

$$i + (h+i_1)\theta < k + k_1\theta < j + (h+j_1)\theta. \tag{10}$$

Let m be large enough that

$$i + i_1\theta < \frac{p}{m} < \frac{p+1}{m} < j + j_1\theta$$

for some $p \in N$. To prove (10), it suffices, on taking $k_1 = 1$, to find h and k such that

$$\frac{p}{m} + h\theta < k + \theta < \frac{p+1}{m} + h\theta. \tag{11}$$

Putting $H = h - 1$, we express (11) as

$$\frac{p}{m} + H\theta < k < \frac{p+1}{m} + H\theta.$$

That is, we seek H for

$$\frac{p}{m} + H\theta < \left\lfloor \frac{p+1}{m} + H\theta \right\rfloor,$$

or, equivalently,

$$\frac{p}{m} + H\theta < \frac{p+1}{m} + H\theta - \{\frac{p+1}{m} + H\theta\},$$

where { } means fractional part. This last inequality is equivalent to

$$\{\frac{p+1}{m} + H\theta\} < \frac{1}{m}. \tag{12}$$

To find such an $H \in N$, we start with the fact that the set $\{H\theta\}$, as H ranges through N, is topologically dense. Thus, in case $(p+1)/m \in N$, we can and do choose H so that $\{H\theta\} < 1/m$, and (12) holds. If $(p+1)/m \notin N$, we can and do choose H so that

$$1 - \{\frac{p+1}{m}\} < \{H\theta\} < 1 - \{\frac{p+1}{m}\} + \frac{1}{m}, \tag{13}$$

so that

$$\{\frac{p+1}{m} + H\theta\} = \{\frac{p+1}{m}\} + \{H\theta\} - 1,$$

which with (13) implies (12). Thus, (D2) holds. One can similarly prove that (D3) holds, so that $(c_n(\theta))$ is a dense fractal sequence.

Example 21. Here, initial terms 1,2 are isolated by 3,4 to form 1423, and then 1,4,2,3 are isolated by 5,6,7,8 to form 18472635, and so on. Concatenating these segments gives

$$(1,1,2,1,2,3,1,3,2,4,1,4,2,3,5,1,4,2,6,3,5,1,4,7,2,6,3,\ldots) = \text{A132223}.$$

The corresponding dense normalized fractal sequence is

$$(1,1,2,1,2,3,1,4,2,3,1,4,2,3,5,1,4,2,6,3,5,1,4,2,6,4,1,4,7,\ldots) = \text{A132224},$$

with placement sequence

$$(1,2,3,2,5,4,3,2,9,8,7,6,5,4,3,2,17,16,15,14,13,12,11,10,9,\ldots) = \text{A132226},$$

formed by concatenating segments $12, 32, 5432, 98765432, \ldots$, where each new segment s_h after the segment 32 is the concatenaton $s_{h-1}s_{h-2}$.

Example 22. Suppose θ is a positive irrational number. For $n \in N$, arrange in increasing order the fractional parts $\{h\theta\}$ for $h = 1, 2, \ldots, n$:

$$\{h_1\theta\} < \{h_2\theta\} < \ \ldots \ < \{h_n\theta\}. \tag{14}$$

From (14), form the segment $(h_1, h_2, \ldots, h_n)$, a permutation of $(1, 2, \ldots, n)$, and proceed as in Section 8 to get a fractal sequence that represents a dense para-sequence. For $\theta = \sqrt{2}$, the sequence is A054073, with placement sequence A054072. For $\theta = (1+\sqrt{5})/2$, the sequence is A054065, for which the placement sequence is John Conway's "left budding sequence":

$$(1, 1, 3, 2, 1, 5, 3, 8, 5, 2, 9, 5, 1, \ldots) = \text{A019587}.$$

The dense normalized fractal sequence obtained from A054065 is

$$(1, 1, 2, 1, 3, 2, 1, 3, 2, 4, 1, 3, 5, 2, 4, 1, 6, 3, 5, 2, 4, 1, 6, 3, 5, 2, 7, 4, \ldots) = \text{A132283},$$

with associated interspersion

$$\begin{pmatrix} 1 & 2 & 4 & 7 & 11 & \cdots \\ 3 & 6 & 9 & 14 & 20 & \cdots \\ 5 & 8 & 12 & 18 & 24 & \cdots \\ 10 & 15 & 21 & 28 & 36 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \text{A132284}.$$

10 Para-sequences and Complementary Pairs of Sequences

A para-sequence f (or $\prec$) is a *para-permutation of* N, in the sense that each element of N occurs exactly once in f. Accordingly, if $n \in N$, then the numbers m satisfying $m \preceq n$ taken together, and those satisfying $m \succ n$ taken together, comprise a partition of N. Taking these numbers as ordered by $\prec$, those to the right of n form a sequence, and those to the left, arranged in reverse order, form a sequence. The two sequences form a pair of complementary sequences which we refer to as the *left* and *right* complementary sequences for the *cut-point* n.

Example 23. The left and right complementary sequences for the cut-point 1 in the para-sequence in Example 19 are

$$(1, 2, 4, 5, 8, 9, 10, 11, \ldots) = A004754 \quad \text{and} \quad (3, 6, 7, 12, 13, 14, 15, \ldots) = A004760.$$

The terms of A004754 have base-2 representations starting with 10, and the terms of A004760 have representations starting with 11.

Conversely, any two complementary sequences can be put together to form a para-permutation of N. For example, using the lower Wythoff sequence, A000201, and upper Wythoff sequence, A001950, as left and right sequences, we have as a para-permutation of N the "limit of the expanding segment"

$$(\ldots, 14, 12, 11, 9, 8, 6, 4, 3, 1, 2, 5, 7, 10, 13, 15, \ldots).$$

Of course, such a para-sequence can be represented as a sequence in the way already described, but it can also representated as a sequence in another way, by jumping back and forth across cut-point like this:

$$(1, 2, 3, 5, 4, 7, 6, 10, 8, 13, 9, 15, 11, \ldots) = A072061.$$

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2000 Mathematics Subject Classification: Primary 11B85

Keywords: Cantor fractal sequence, dense fractal sequence, dispersion, Farey fractal sequence, fractal sequence, infinitive sequence, interspersion, normalized fractal sequence, para-sequence, proper self-containing sequence, signature sequence

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(Concerned with sequences A000201, A001511, A001950, A002260, A003602, A003603, A004711, A004736, A004754, A004760, A007336, A019587, A022446, A022447, A023115, A023133, A054065, A054572, A054573, A054582, A072061, A088370, A108712, A087467, A120873, A122196, A125158, A131966, A131967, A131968, A131987, A132223, A132224, A132226, A132283, A132284)