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ABSTRACT. The manuscript details Alekseyev's bijection between splitting $[n]$ into sets of lists with odd numbers of elements and coverings of the labeled graph with two vertices and n edges in [1, A131518].

1. Nomenclature

Sequence [1, A088009] counts the partitions of $[n]$ into sets of non-overlapping lists of odd length. Lists are defined as usual in programming languages, which means they are 'ordered' tuples of integers. Each of these sets has a representation

$$\{[n_{1,1}, n_{1,2}, \ldots, n_{1,l_1}], [n_{2,1}, n_{2,2}, \ldots, n_{2,l_2}], .., [n_{l,1}, n_{l,2}], \ldots\} \tag{1}$$

where $n_{l,i}$ is the i-th element in list l, all elements $n_{l,i}$ are distinct, all lengths l_i are odd, and the union of all $n_{l,i}$ is $\{1, 2, \ldots n\}$.

Example 1. *The sets of odd lists of* $[3]$ *are* $\{[1], [2], [3]\}$, $\{[1, 2, 3]\}$, $\{[1, 3, 2]\}$, $\{[2, 1, 3]\}$, $\{[2, 3, 1]\}$, $\{[3, 1, 2]\}$, $\{[3, 2, 1]\}$.

Remark 1. *To construct these lists on a computer, one would start with an outer loop of all partitions of* n *into odd parts* [1, A000009], $n = p_1^{f_1}; p_2^{f_2}; \ldots$, *where part* p_i *occurs* f_i *times, and all* p_i *are odd. The one would consider all* $n!$ *lists of* $[n]$, *chop them left-to-right according to the number of parts, and reduce the counts by the number of sub-list-sets considered equivalent.*

Sequence [1, A131518] considers a connected multi-graph with two labeled vertices v_1, v_2 connected by n labeled edges $e_1, e_2, \ldots e_n$ where each edge connects v_1 and v_2. (There are no loops.) The sequence counts coverings by the graph by a number of dipaths in the following sense:

- The union of 2 paths is not a path. (I.e., the last vertex of a path is not the first vertex of another path.)
- Covering property: Each edge is member of exactly one path.
- Paths are along digraphs; so if edge number i is traveled from v_1 to v_2 it is considered different from being traveled from v_2 to v_1.

The edges are not oriented (arcs) a priori and can be traveled in both direction. (The graph is not fixed in that sense but we are considering graphs in all 2^n sets of orientations of the labeled edges.) Each covering has a representation of c paths which can be written as a succession of a list of vertices $v_{1,\ldots}$ and edges $e_{1,\cdot}$ of the

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first path, a list of vertices $v_{2,\cdots}$ and edges $e_{2,\cdot}$ of the second path, etc

$$\{v_{1,i}(e_{1,j})v_{1,i+1}(e_{1,j+1})\cdots v_{1,i+l_1};\\ v_{2,i}(e_{2,j})v_{1,i+1}(e_{2,j+1})\cdots v_{2,i+l_1};\\ \ldots; v_{c,i}(e_{c,j})v_{c,i+1}(e_{2,j+1})\cdots v_{c,i+l_1}\}. \tag{2}$$

The orientation of the edges is implicit in the notation by the indices of the vertices left and right to it.

Example 2. *The two coverings for $n = 1$ are $\{v_1(e_1)v_2\}$, $\{v_2(e_1)v_1\}$.*

Example 3. *The 6 coverings for $n = 2$ are 2 coverings with $c = 2$ paths $\{v_1(e_1)v_2; v_1(e_2)v_2\}$, $\{v_2(e_1)v_1; v_2(e_2)v_1\}$, and 4 coverings with $c = 1$ path $\{v_1(e_1)v_2(e_2)v_1\}$, $\{v_1(e_2)v_2(e_1)v_1\}$, $\{v_2(e_1)v_1(e_2)v_2\}$, $\{v_2(e_2)v_1(e_1)v_2\}$.*

2. Bijection

Aleksejev's constructive bijection to construct the coverings from the lists of odd length works as follows:

(1) For each of the sets of lists in (1) start a path at v_1 and consider the (distinct) $n_{1,i}$ of the first list as the list of edges to follow in the path, start another path at v_1 and consider the (distinct) $n_{2,i}$ of the second list as the list of edges to follow in the second path and so on. Because all the lists in (1) have odd length, all these paths end at v_2 so the union of these paths is never a path. The number of paths in a covering is the number of lists in the list of $[n]$, i.e., the number of parts in the odd partition of n.

Remark 2. *The simplest illustration is to regard a path as a list of strokes with a pen along edges; the restriction on the union of paths means that two paths can only be painted by lifting the pen once in between.*

(2) Construct another set of paths by taking the same set of lists again but start each path now at v_2 instead of v_1.

(3) For even n there are coverings which are missing in the previous two algorithms: Coverings where one path starts and ends at v_1 along all edges, or one path starts and ends at v_2 along all edges, or two paths of even lengths, one starting at v_1 the other at v_2. These coverings are constructed by starting from all $n!$ lists $[n_1, n_2, n_3, \ldots n_n]$ of $[n]$ in three different ways;
  (a) One covering is starting at v_1 and considering each of these lists as a sequence of edge indices (and ending at v_1 because n is even).
  (b) One covering is starting at v_2 and considering each of these lists as a sequence of edge indices (and ending at v_2 because n is even).
  (c) $n/2 - 1$ paths considering a list 'interrupted' once after either n_2 or n_4 or n_6 or… n_{n-2} (which creates two sublists of even length), and interpreting the interruption as a 'lifting of the pen', i.e., starting the second path at the opposite vertex of the first path. The first sublist is the edge index list of a path starting at v_1, the second sublist is the edge index list of a path starting at v_2.

  Remark 3. *We avoid over-counting. Consider $n = 6$ and the lists* $[1, 2, 3, 4, 5, 6], [1, 2, 3, 4, 6, 5], \ldots [6, 5, 4, 3, 2, 1]$ *and 'complementary' interruptions with swapped sublists like* $[1, 2|4, 3, 5, 6]$ *and* $[4, 3, 5, 6|1, 2]$

in two different lists. The first interruption started at v_1 and the second list started at v_2 are the same set of two paths, and to be counted only once.

The number of additional paths generated for even n this way is $n![1+1+(n/2-1)] = n!/(1+n/2)$.

In summary, there are A088009 paths generated in (1), A088009 paths generated in (2), and $n!(1+n/2)$ paths generated in (3), which is Alekseyev's formula.

References

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