Products of Consecutive-Integer Ratios

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Consider the random product

$$P(N)=\prod_{n=1}^{N}\left(\frac{n}{n+1}\right)^{\varepsilon_n}=\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)^{-\varepsilon_n}$$

where ε_1, ε_2, ..., ε_N are independent variables satisfying $P(\varepsilon_n=1)=P(\varepsilon_n=-1)=1/2$ for each n. The maximum value of $P(N)$ is $N+1$, which occurs if and only if all ε_n are -1. The minimum value of $P(N)$ is $1/(N+1)$, which occurs if and only if all ε_n are 1. We are interested in the average behavior of $P(N)$ and it makes sense to examine $\ln(P(N))$ henceforth (with extreme values $-\ln(N+1)$ and $\ln(N+1)$ symmetric about the origin).

Before continuing, let us mention the random sum

$$S(N)=\sum_{n=1}^{N}\frac{\varepsilon_n}{n}$$

which converges almost surely [1, 2]. The maximum value of $S(N)$ diverges to ∞ as $N\to\infty$ and the minimum value of $S(N)$ diverges to $-\infty$. Clearly $\mathrm{E}(S(N))=0$ and

$$\mathrm{Var}(S(N))=\sum_{n=1}^{N}\frac{1}{n^2}\to\frac{\pi^2}{6}$$

as $N\to\infty$. It is perhaps surprising that $\mathrm{Var}(S(N))$ is finite. Define $\theta_n=-1$ if $n\equiv 0\bmod 3$ and $\theta_n=1$ otherwise; define $\omega_n=-1$ if $n\equiv 2,3\bmod 4$ and $\omega_n=1$ otherwise. On the one hand [3],

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln(2),\qquad \sum_{n=1}^{\infty}\frac{\omega_n}{n}=\frac{\pi}{4}-\frac{1}{2}\ln(2);$$

on the other hand [4],

$$\sum_{n=1}^{N}\frac{1}{n}\sim\ln(N)+\gamma,\qquad \sum_{n=1}^{N}\frac{\theta_n}{n}\sim\frac{1}{3}\ln(N)+\frac{2}{3}\ln(3)+\frac{1}{3}\gamma$$

where γ is the Euler-Mascheroni constant [5].

Returning to the product $P(N)$, we have $\mathrm{E}(\ln(P(N))) = 0$ and

$$\mathrm{Var}(\ln(P(N))) = \sum_{n=1}^{N} \ln\left(\frac{n}{n+1}\right)^2 \to 0.977189...$$

as $N \to \infty$. No closed-form expression for this expression is known. Again, it is perhaps surprising that $\mathrm{Var}(\ln(P(N)))$ is finite. By Wallis' formula [6], we have

$$\prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{(-1)^{n+1}} = \frac{1}{2}\frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\frac{7}{6}\frac{7}{8}\cdots = \frac{2}{\pi}$$

but as before an unbalanced distribution of $+1$ and -1 exponents leads to divergence (to either ∞ or 0).

Here is a far more difficult problem. Let $a(N)$ and $b(N)$ denote the numerator and denominator of $P(N)$, expressed in lowest terms. Rather than maximizing $P(N)$ for fixed N as previously, consider instead maximizing $a(N)$. Note that, by changing each ε_k to $-\varepsilon_k$, the maximum value of $b(N)$ is equal to the maximum value of $a(N)$. Hence we lose nothing by studying only numerators in the following.

Let $A(N)$ denote the maximum value of $a(N)$. See Table 1 for sample values [7]. For example, when $N = 6$,

$$\text{the numerator of} \quad \left(\tfrac{1}{2}\right)^{-1} \tfrac{2}{3} \left(\tfrac{3}{4}\right)^{-1} \tfrac{4}{5} \left(\tfrac{5}{6}\right)^{-1} \tfrac{6}{7} \quad \text{is } 2^8$$

whereas

$$\text{the numerator of} \quad \left(\tfrac{1}{2}\right)^{-1} \left(\tfrac{2}{3}\right)^{-1} \left(\tfrac{3}{4}\right)^{-1} \tfrac{4}{5} \left(\tfrac{5}{6}\right)^{-1} \tfrac{6}{7} \quad \text{is } 2^6 3^2;$$

hence $A(6) = 576$. Nicolas [8] and de la Bretèche, Pomerance & Tenenbaum [9] proved that

$$0.107 < \liminf_{N\to\infty} \frac{1}{N\ln(N)} \ln(A(N)) \le \limsup_{N\to\infty} \frac{1}{N\ln(N)} \ln(A(N)) \le \frac{2}{3} < 0.667.$$

At the end of [9], the lower bound was improved to 0.112 (due to Fouvry). We wonder whether the limit supremum is equal to the limit infimum and, if so, what the limiting value might be.

Table 1 *Sample Values of Maximum Numerator* $A(N)$ *and of* $\ln(A(N))/(N\ln(N))$

N	1	2	3	4	5	6
$A(N)$	2	4	16	64	128	576
$\frac{\ln(A(N))}{N\ln(N)}$		1.0000	0.8407	0.7500	0.6031	0.5909
N	7	8	9	10	11	12
$A(N)$	4608	16384	64000	640000	2560000	10240000
$\frac{\ln(A(N))}{N\ln(N)}$	0.6195	0.5833	0.5596	0.5806	0.5592	0.5414

0.1. Highly Composite Numbers. A positive integer n is **highly composite** if, for all $m < n$, we have $d(m) < d(n)$, where $d(k)$ denotes the number of distinct divisors of k. The integer n is also called a d-**champion**. It is known that

$$\left|\{n \leq N : n \text{ is highly composite}\}\right| = O\left(\ln(N)^{1.71}\right)$$

as $N \to \infty$, and conjectured that 1.71 can be replaced by any constant $c > \ln(30)/\ln(16) = 1.2267...$ [10].

A positive integer n is **superior highly composite** if there exists $\delta > 0$ such that, for all positive integers m, we have $d(m)/m^{\delta} \leq d(n)/n^{\delta}$. It is known that

$$\left|\{n \leq N : n \text{ is superior highly composite}\}\right| \sim \ln(N)$$

as $N \to \infty$. While these asymptotics are well-understood, those for the quotient of two consecutive highly composite numbers are not.

Define

$$\lambda = \limsup_{N \to \infty} \frac{1}{N \ln(N)} \ln(A(N))$$

where $A(N)$ is as before. If M is a sufficiently large superior highly composite number and M' is the highly composite number following M, then [10]

$$\frac{M'}{M} \geq 1 + \frac{1}{\ln(M)^{\kappa}}$$

for any constant $\kappa > \lambda/\ln(2)$. Since we know $\lambda \leq 2/3$, it follows that the exponent $2/(3\ln(2)) = 0.961796...$ works. A sharper upper bound on λ (for example, $\lambda \leq 3/5$ or even $\lambda \leq 1/2$) would be very helpful.

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