Generating functions for circular variants of squares of binomial coefficients

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In this note, we prove two formulae for circular variants of squares of binomial coefficients that appear in sequence A123610. For $n > 0$ and $0 \le k \le n$, the elements of this (triangular) sequence are defined by

$$T(n,k) = \frac{1}{n} \sum_{d \mid \gcd(n,k)} \phi(d) \binom{\frac{n}{d}}{\frac{k}{d}}^2.$$

Here $\phi(d)$ is Euler's totient function at positive integer d. (The sequence A123610 actually starts with $T(0,0) = 1$, but we shall ignore this term.)

First, we prove that the bivariate g.f. of the numbers $T(n,k)$ is

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n} T(n,k)\, x^n y^k = -\sum_{s=1}^{\infty} \frac{\phi(s)}{s} \log f(x^s, y^s), \tag{1}$$

where

$$f(x,y) = \frac{\sqrt{g(x,y)} + 1 - (1+y)x}{2} \quad \text{and} \quad g(x,y) = 1 - 2(1+y)x + (1-y)^2 x^2. \tag{2}$$

For given $n > 0$, we also prove that the univariate g.f. of the numbers $T(n,k)$ is given by

$$\sum_{k=0}^{\infty} T(n,k) y^k = \frac{1}{n} \sum_{d \mid n} \phi(d) R\left(\frac{n}{d}, y^d\right), \tag{3}$$

where

$$R(m,y) = [z^m](1 + (1+y)z + yz^2)^m = \frac{1}{m!} \frac{d^m}{dz^m} \left((1 + (1+y)z + yz^2)^m\right). \tag{4}$$

Throughout this note, we assume $T(n,k) = 0 = \binom{n}{k}$ for $0 \le n < k$.

Proof. Since

$$\lim_{(x,y)\to(0,0)} g(x,y) = 1 = \lim_{(x,y)\to(0,0)} f(x,y),$$

there is $\delta \in (0,1)$ such that $g(x,y) > 0$ and $f(x,y) > 0$ for all $(x,y) \in (-\delta,\delta)\times(-\delta,\delta)$. All the calculations that follow are valid (at least) for this rectangle.

To prove equation (1), we use the following formula by E. Deutsch and P. Bala,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}^2 x^n y^k = \frac{1}{\sqrt{g(x,y)}}, \tag{5}$$

where $g(x,y)$ is given by the second equation in (2). This formula appears at two different places in the documentation of sequence A008459. Equation (5) implies

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}^2 x^{n-1} y^k = \frac{1}{x}\left(\frac{1}{\sqrt{g(x,y)}} - 1\right).$$

It follows that

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}^2 \frac{x^n}{n} y^k = \int_0^x \frac{1}{w}\left(\frac{1}{\sqrt{g(w,y)}} - 1\right) dw. \tag{6}$$

Note that for each $y \in (-\delta,\delta)$,

$$\lim_{w\to 0}\left(\frac{1}{\sqrt{g(w,y)}} - 1\right) = 0, \quad \text{and thus,} \quad \lim_{w\to 0}\frac{1}{w}\left(\frac{1}{\sqrt{g(w,y)}} - 1\right) = 1 + y > 0.$$

Hence, there is a removable singularity at $w = 0$ for the integrand in the right-hand side of equation (6).

We claim that, for each $y \in (-\delta,\delta)$,

$$\int \frac{1}{w}\left(\frac{1}{\sqrt{g(w,y)}} - 1\right) dw = -\log f(w,y) + C \quad (\text{for } w \in (-\delta,\delta)). \tag{7}$$

We differentiate the RHS of equation (7) w.r.t. w:

$$-\frac{\partial}{\partial w}\log f(w,y) = \frac{-\frac{\frac{\partial}{\partial w}g(w,y)}{2\sqrt{g(w,y)}} + (1+y)}{\sqrt{g(w,y)} + 1 - (1+y)w} = \frac{(1+y) - (1-y)^2 w + (1+y)\sqrt{g(w,y)}}{\sqrt{g(w,y)}\left(\sqrt{g(w,y)} + 1 - (1+y)w\right)}.$$

To finish the proof of equation (7), we need to prove that

$$\frac{(1+y) - (1-y)^2 w + (1+y)\sqrt{g(w,y)}}{\sqrt{g(w,y)} + 1 - (1+y)w} = \frac{1 - \sqrt{g(w,y)}}{w}.$$

This can be achieved by cross-multiplication of the two fractions.

Since $f(0,y)=1$ for each $y\in(-\delta,\delta)$, equations (6) and (7) imply

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}^2\frac{x^n}{n}y^k=-\log f(x,y).$$

We then have

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}T(n,k)\,x^ny^k=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n}\sum_{d|\gcd(n,k)}\phi(d)\binom{\frac{n}{d}}{\frac{k}{d}}^2x^ny^k.$$

Letting $n^*=n/d$ and $k^*=k/d$, we then get:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}T(n,k)\,x^ny^k=\sum_{d=1}^{\infty}\frac{\phi(d)}{d}\sum_{n^*=1}^{\infty}\sum_{k^*=0}^{\infty}\binom{n^*}{k^*}^2\frac{(x^d)^{n^*}}{n^*}(y^d)^{k^*}=-\sum_{d=1}^{\infty}\frac{\phi(d)}{d}\log f(x^d,y^d).$$

This completes the proof of equation (1).

To prove equation (3), we use an equation by P. Bala, which is stated in the documentation of sequence A008459. For each $n>0$,

$$\sum_{k=0}^{\infty}\binom{n}{k}^2y^k=R(n,y),$$

where $R(n,y)$ is given by equations (4). We then have

$$\sum_{k=0}^{\infty}T(n,k)\,y^k=\sum_{k=0}^{\infty}\frac{1}{n}\sum_{d|\gcd(n,k)}\phi(d)\binom{\frac{n}{d}}{\frac{k}{d}}^2y^k=\frac{1}{n}\sum_{d|n}\phi(d)\sum_{k^*=0}^{\infty}\binom{\frac{n}{d}}{k^*}^2(y^d)^{k^*}=\frac{1}{n}\sum_{d|n}\phi(d)R\left(\frac{n}{d},y^d\right).$$

This completes the proof of equation (3) as well. □