We get from first principles that the count of directed labeled multigraphs including isolated vertices on m vertices and n arcs is

$$S_{n,m} = [z^n]\frac{1}{(1-z)^{m(m-1)}} = \binom{n+m(m-1)-1}{n}.$$

Here we are simply connecting $m(m-1)$ ordered pairs of vertices by some number of edges. (Ordered pairs because these multigraphs are directed.) We then obtain for the count $T_{n,m}$ where there are no isolated vertices

$$\sum_{q=0}^{m}\binom{m}{q}T_{n,m-q} = \sum_{q=0}^{m}\binom{m}{q}T_{n,q} = S_{n,m}.$$

Here we have by inspection that for $T_{n,m}$ we must have $m \le 2n.$ Inverting the binomial transform we find

$$T_{n,m} = \sum_{q=0}^{m}\binom{m}{q}(-1)^{m-q}S_{n,q}.$$

We thus have for the desired quantity Q_n that it is given by

$$Q_n = \sum_{m=0}^{2n}\sum_{q=0}^{m}\binom{m}{q}(-1)^{m-q}S_{n,q}.$$

The goal now becomes to simplify this expression to get the formula given at this OEIS entry, which lists the matching values for Q_n

$$2, 27, 572, 16787, 631362, 28980861, 1570956872, 98212870233, \ldots$$

We find

$$\sum_{m=0}^{2n}\sum_{q=0}^{m}\binom{m}{q}(-1)^{m-q}\binom{n+q(q-1)-1}{n} \\ = \sum_{q=0}^{2n}(-1)^q\binom{n+q(q-1)-1}{n}\sum_{m=q}^{2n}\binom{m}{q}(-1)^m.$$

We get for the inner sum,

$$\sum_{m=0}^{2n-q}\binom{2n-m}{q}(-1)^{2n-m} = [z^{2n-q}](1+z)^{2n}\sum_{m=0}^{2n-q}(-1)^m\frac{z^m}{(1+z)^m} \\ = [z^{2n-q}](1+z)^{2n}\frac{1-(-1)^{2n+1-q}(z/(1+z))^{2n+1-q}}{1+z/(1+z)} \\ = [z^{2n-q}](1+z)^{2n+1}\frac{1-(-1)^{2n+1-q}(z/(1+z))^{2n+1-q}}{1+2z}.$$

Here the coefficient extractor cancels the second summand in the fraction and we get

$$[z^{2n-q}](1+z)^{2n+1}\frac{1}{1+2z}.$$

Making the substitution

$$Q_n = [z^{2n}](1+z)^{2n+1}\frac{1}{1+2z}\sum_{q=0}^{2n}(-1)^q\binom{n+q(q-1)-1}{n}z^q.$$

The contribution from z is

$$\frac{1}{2}\operatorname*{res}_{z}\frac{1}{z^{2n+1-q}}(1+z)^{2n+1}\frac{1}{z+1/2}.$$

Residues sum to zero and we may evaluate using minus the residue at $z=-1/2$ and $z=\infty$. We get for the former,

$$-\frac{1}{2}(-1)^{2n+1-q}2^{2n+1-q}\frac{1}{2^{2n+1}} = (-1)^q\frac{1}{2^{q+1}}$$

which gives the first piece

$$\sum_{q=0}^{2n}\binom{n+q(q-1)-1}{n}\frac{1}{2^{q+1}}.$$

With minus the residue at infinity we obtain

$$\begin{aligned}&\frac{1}{2}\operatorname*{res}_{z}\frac{1}{z^2}z^{2n+1-q}(1+1/z)^{2n+1}\frac{1}{1/z+1/2}\\ &= \frac{1}{2}\operatorname*{res}_{z}\frac{1}{z^{q+1}}(1+z)^{2n+1}\frac{1}{1+z/2}\\ &= \sum_{p=0}^{q}\binom{2n+1}{p}(-1)^{q-p}\frac{1}{2^{q+1-p}}.\end{aligned}$$

Substitute into the sum in q to get

$$\sum_{q=0}^{2n}\binom{n+q(q-1)-1}{n}\frac{1}{2^{q+1}}\sum_{p=0}^{q}\binom{2n+1}{p}(-1)^p 2^p.$$

Now when $q \ge 2n+1$ the inner sum evaluates to -1 so this second piece is

$$\begin{aligned}&\sum_{q\ge 0}\binom{n+q(q-1)-1}{n}\frac{1}{2^{q+1}}\sum_{p=0}^{q}\binom{2n+1}{p}(-1)^p 2^p\\ &+\sum_{q\ge 2n+1}\binom{n+q(q-1)-1}{n}\frac{1}{2^{q+1}}.\end{aligned}$$

This observation also applies concerning convergence of the first sum which starts with a finite segment up to $q=2n+1$ and thereafter the minus one appears and the outer sum converges. Adding in the first piece to the second we get the desired

$$\boxed{Q_n = \sum_{q\ge 0}\binom{n+q(q-1)-1}{n}\frac{1}{2^{q+1}}.}$$

where it remains to show that

$$\sum_{q\ge 0} {n+q(q-1)-1\choose n} \frac{1}{2^{q+1}} \sum_{p=0}^q {2n+1\choose p} (-1)^p 2^p = 0.$$

This is

$$-Q_n - \sum_{q\ge 0} {n+q(q-1)-1\choose n} \frac{1}{2^{q+1}} \sum_{p=q+1}^{2n+1} {2n+1\choose p} (-1)^p 2^p.$$

The inner sum is zero when $q > 2n$ so we get

$$-Q_n - \sum_{q=0}^{2n} {n+q(q-1)-1\choose n} \frac{1}{2^{q+1}} \sum_{p=q+1}^{2n+1} {2n+1\choose p} (-1)^p 2^p \\ = -Q_n + \sum_{q=0}^{2n} {n+q(q-1)-1\choose n} \frac{1}{2^{q+1}} 2^{2n+1} \sum_{p=0}^{2n-q} {2n+1\choose p} (-1)^{-p} 2^{-p}.$$

The inner term is

$$2^{2n-q} [z^{2n-q}] \frac{1}{1-z} (1-z/2)^{2n+1} = [z^{2n-q}] \frac{1}{1-2z} (1-z)^{2n+1} \\ = (-1)^q [z^{2n-q}] \frac{1}{1+2z} (1+z)^{2n+1}.$$

We have obtained $-Q_n + Q_n = 0$ as required which concludes the argument.

This was [math.stackexchange.com problem 5027154](https://math.stackexchange.com/questions/5027154/).