ON THE GENERATING FUNCTION OF A119574

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For a nonnegative integer n, let

$$a_n = \frac{\binom{2n}{n}(n+2)^2}{n+1}$$

and let $f(x)$ be the corresponding generating function. Then

$$f(x) = \frac{(1-4x)\sqrt{1-4x} - 36x^2 + 14x - 1}{2x(\sqrt{1-4x})^3},$$

confirming the conjecture in A119574. Indeed, using that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0}\binom{2n}{n}x^n,$$

we have

$$\begin{aligned}
\sum_{n\geq 0}\frac{\binom{2n}{n}(n+2)^2}{n+1}x^n &= \sum_{n\geq 0}\frac{\binom{2n}{n}(n+1+1)^2}{n+1}x^n \\
&= \sum_{n\geq 0}\frac{\binom{2n}{n}\left((n+1)^2+2(n+1)+1\right)}{n+1}x^n \\
&= 3\sum_{n\geq 0}\binom{2n}{n}x^n + \sum_{n\geq 0}\binom{2n}{n}nx^n + \sum_{n\geq 0}\frac{\binom{2n}{n}}{n+1}x^n \\
&= 3\sum_{n\geq 0}\binom{2n}{n}x^n + x\left(\sum_{n\geq 0}\binom{2n}{n}x^n - 1\right)' + \frac{1}{x}\int_0^x\sum_{n\geq 0}\binom{2n}{n}t^n dt \\
&= \frac{3}{\sqrt{1-4x}} + \frac{2x}{\sqrt{1-4x}^3} - \frac{\sqrt{1-4x}}{2x} + \frac{1}{2x} \\
&= \frac{(1-4x)\sqrt{1-4x} - 36x^2 + 14x - 1}{2x(\sqrt{1-4x})^3}
\end{aligned}$$

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References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., https://oeis.org.