A note on the cylinder poset

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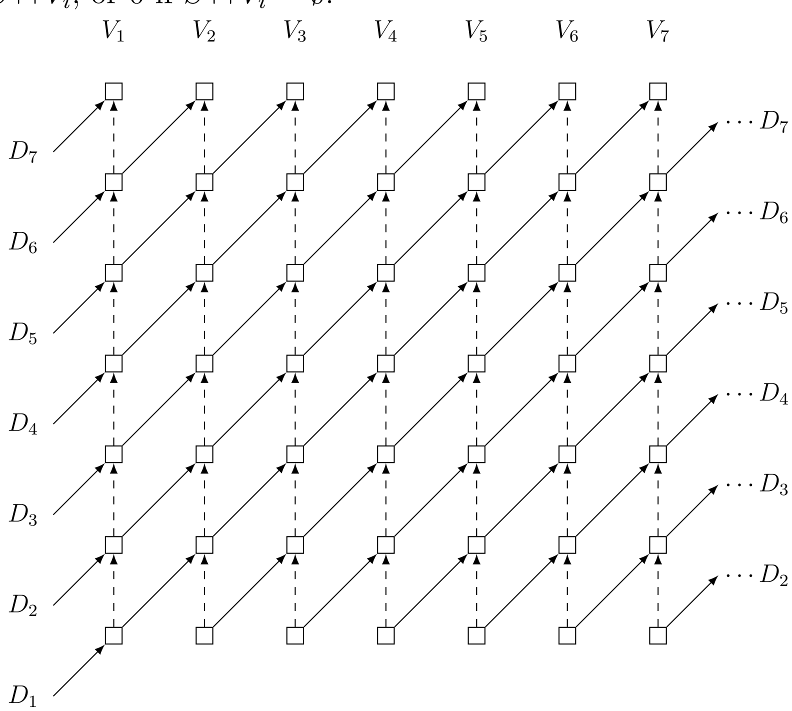
We define Cyl_n, the *cylinder poset* of order n, as follows: its elements are pairs $\{(i,j) | i,j \in [n]\}$, and its covering relations take two forms:

- $(i,j) \succ (i,j-1)$ for $i \in [n]$, $j \in [2,n]$. These relations define the "vertical chains" $V_i = \{(i,j)\}_{j\in[n]}$.
- $(i,j) \succ (i-1,j-1)$ for $i \in [n]$, $j \in [2,n]$ and $(1,j) \succ (n,j-1)$ for $j \in [2,n]$. These relations define the "diagonal chains" $D_k = \{(i, k+(i-1) \mod n)\}_{i\in[n]}$, where $\mod$ "wraps around" to 1 instead of to 0.

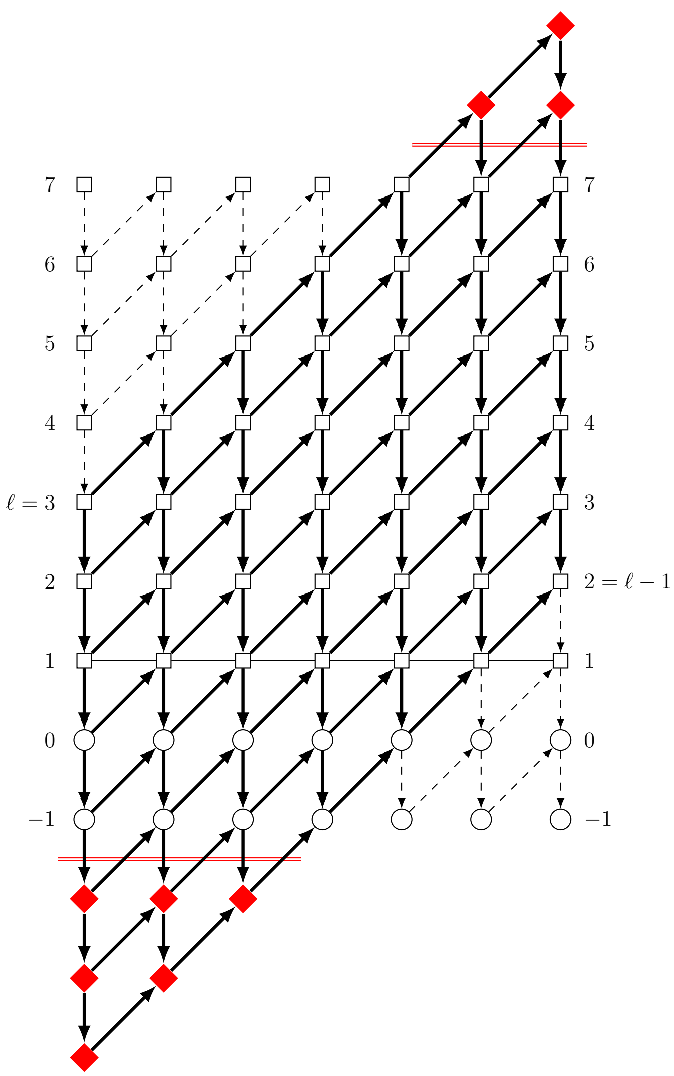
It's called a cylinder poset because the diagonal grids "jump" from one side to the other as if the poset were on the surface of a cylinder. For a node $n = (i,j)$, we refer to j as the *height* of n. For a set S of vertices of Cyl_n, we define the *height of S along V_i* as the maximum height of an element of $S \cap V_i$, or 0 if $S \cap V_i = \emptyset$.

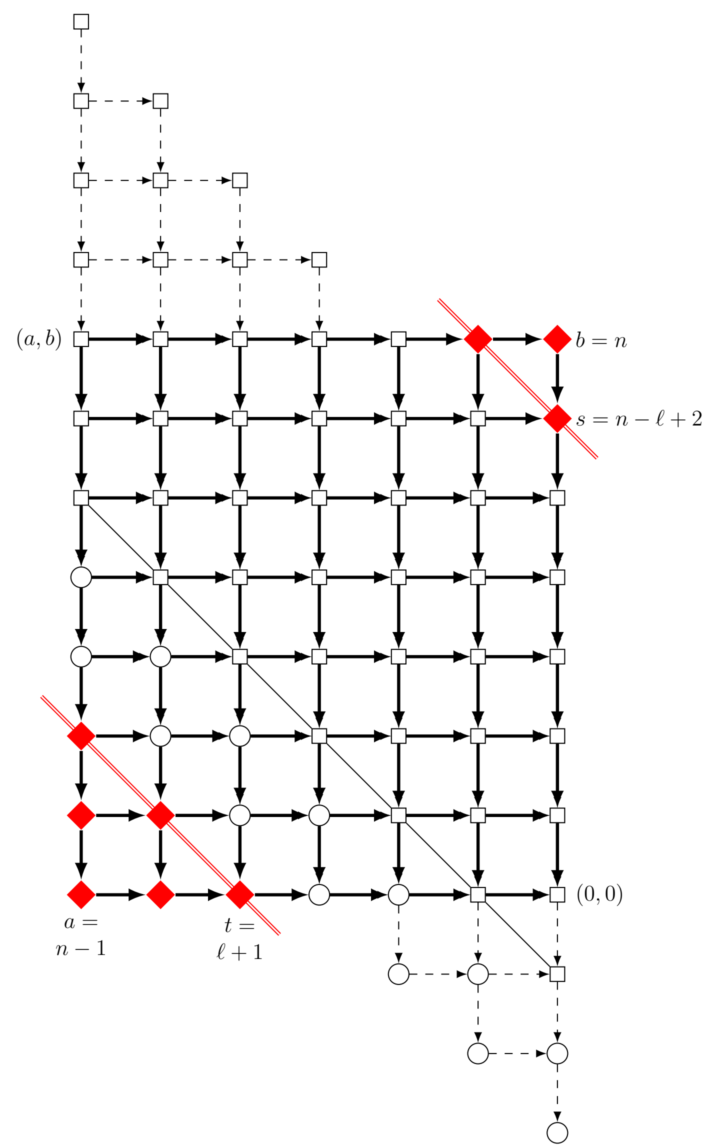


The main result of this note is to count the number of downward closed sets in Cyl_n using known path-counting results.

First, it's helpful to observe that there's a bijection between the downward closed sets of Cyl_n and the set of sequences $(a_1, \ldots, a_n)$ such that $a_i \in [0, n]$ for $i \in [n]$, $a_{i+1} \leq a_i + 1$ for $i \in [n-1]$, and $a_1 \leq a_n + 1$. Given a downward closed set $S \subseteq \mathrm{Cyl}_n$, the bijection is given by letting a_i be the height of S along V_i. The condition that $a_{i+1 \bmod n} \leq a_i + 1$ is equivalent to being downward closed along the diagonal chains $\{D_k\}$.

Now, fix ℓ as the height of S along V_1. Note that taking $\ell = 0, 1, \ldots, n$ partitions the collection of downward closed sets. Now we identify the downward closed set S with its upward boundary in the Hasse diagram of Cyl_n. This upward boundary can then be uniquely identified with a lattice path starting at $(1, \ell)$ where at each node you can take an upward edge along a diagonal chain D_k or a downward edge along a vertical chain V_i. This ensures that S is downward closed along each V_i and for the "internal" diagonal relations, i.e. those of the form $(i+1, j+1) \succ (i, j)$. To satisfy the relations $(1, j+1) \succ (n, j)$, we just need the path to terminate at $(n, \ell - 1)$. To properly represent the fact that $S \cap V_i = \emptyset$ for a given i, we draw elements of height 0 and -1, and when $S \cap V_i = \emptyset$ we let the path pass through $(i-1, -1)$ and $(i, 0)$. A path of this form uniquely determines a downward-closed set.





Thus, the problem is reduced to counting lattice paths in a square grid with "missing corners", i.e. those paths which avoid certain "translated diagonals". This problem was clasically solved. For an accessible proof, see [DR15] (`https://oeis.org/A136439/a136439.pdf`), Equation (5).

Theorem 1. *The number of monotonic integer lattice paths from* $(0,0)$ *to* (a,b) *avoiding the lines* $y = x + s$ *and* $y = x - t$ *is equal to*

$$\mathscr{L}(a, b; s, t) = \sum_{k\in\mathbb{Z}} \left[\binom{a+b}{b+k(s+t)} - \binom{a+b}{b+k(s+t)+t} \right]$$

where $\binom{u}{v} = 0$ *for* $v < 0$ *or* $v > u$.

For a fixed n, in the reduction to the above problem $a = n-1$ and $b = n$ are constant. With some careful counting you see that $t = \ell + 1$ and $s = n - \ell + 2$ (The number of "forbidden nodes" in the bottom corner is $(n+1) - (\ell+2) = n - \ell - 1$, so $t = n - 1 - (n - \ell - 1) + 1$. Likewise, there are $(\ell - 1 + n) - n = \ell - 1$ forbidden nodes in the upper corner, so $s = n - (\ell - 1) + 1$.)

Thus, the number of downward closed sets in Cyl_n is exactly

$$\begin{aligned}
\sum_{\ell=0}^{n} \mathscr{L}(n-1, n; n-\ell+2, \ell+1) &= \sum_{\ell=0}^{n} \sum_{k\in\mathbb{Z}} \binom{2n-1}{n+k(n+3)} - \binom{2n-1}{n+\ell+1+k(n+3)} \\
&= \sum_{\ell=0}^{n} \binom{2n-1}{n} - \binom{2n-1}{n+\ell+1} - \binom{2n-1}{\ell-2} \\
&= (n+1)\binom{2n-1}{n} - \sum_{\ell=0}^{n} \binom{2n-1}{n+\ell+1} + \binom{2n-1}{\ell-2} \\
&= (n+1)\binom{2n-1}{n} - \left(-\binom{2n-1}{n-1} - \binom{2n-1}{n} + \sum_{k=0}^{2n-1} \binom{2n-1}{k} \right) \\
&= (n+3)\binom{2n-1}{n} - 2^{2n-1}
\end{aligned}$$

References

[DR15] Nachum Dershowitz and Christian Rinderknecht. The average height of catalan trees by counting lattice paths. *Mathematics Magazine*, 88(3):187–195, 2015.