SUMS OF FIBONACCI NUMBERS INDEXED BY INTEGER PARTS

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ABSTRACT. Consider the integer sequences $(F_{\lfloor \sqrt{n} \rfloor} : n \in \mathbb{N}_0)$ and $(F_{\lfloor \log_2 n \rfloor} : n \in \mathbb{N})$, letting $\lfloor x \rfloor$ denote the integer part of a nonnegative value x, and where F_n denotes the n^{th} Fibonacci number for a nonnegative integer n. We apply an Abel-type summation lemma to prove explicit evaluations for $\sum_{n=1}^{m} F_{\lfloor \sqrt{n} \rfloor}$ and $\sum_{n=1}^{m} F_{\lfloor \log_2 n \rfloor}$, for a natural number m. We then apply this summation lemma to determine an analytical formula for $\sum_{n=1}^{m} F_{\lfloor \frac{n}{s} \rfloor}$, letting s denote a natural number parameter, and we demonstrate how our method may be applied to evaluate sums of the form $\sum_{n=1}^{m} F_{\left\lfloor \frac{\sqrt[r]{n}}{s} \right\rfloor}$ for integers $r \geq 2$ and $s \geq 1$. We also consider the problem of evaluating finite sums of expressions of the form $F_{\left\lfloor \log_2\left(\frac{n}{s}\right) \right\rfloor}$ for a natural number s. Much of our work is is closely connected with evaluations for Fibonacci sums of the form $S(t, m) = \sum_{n=1}^{m} n^t F_n$, where t is a nonnegative integer.

1. Introduction

A large amount of research that has been based on the Fibonacci sequence $(F_n : n \in \mathbb{N}_0)$ has concerned identities for finite summations involving entries in this sequence. This article is inspired by many previous research contributions on finite sums of this form, as in Chu's recent paper [5] on repeated applications of the partial sum operator to the Fibonacci sequence, along with the recent work of Ollerton and Shannon [17] (cf. [2, 12]) on sums of the form

$$S(t, m) = \sum_{n=1}^{m} n^t F_n, \tag{1.1}$$

where t is a nonnegative integer and m denotes a natural number. The finite difference-based approach employed in [2] in the study of summations of the form

$$\sum_{n=1}^{m} n^t F_{n+r}$$

is of relevance to our article, in which we use an Abel-type summation lemma involving difference operators to prove identities for the partial sums of naturally occurring integer sequences defined via the Fibonacci sequence.

In Section 2, we introduce and prove an explicit, non-recursive formula for the partial sums for the sequence

$$\left(F_{\lfloor \sqrt{n} \rfloor} : n \in \mathbb{N}_0 \right), \tag{1.2}$$

letting $\lfloor \cdot \rfloor$ denote the floor function. Then, in Section 3, we use a similar approach to explicitly evaluate the partial sums of

$$\left(F_{\lfloor \log_2(n) \rfloor} : n \in \mathbb{N} \right). \tag{1.3}$$

Then, in Section 4, we apply our Abel summation-based method to evaluate the partial sums of infinite families of sequences of Fibonacci numbers indexed by $\lfloor \cdot \rfloor$ composed with an expression involving a free natural number parameter. In this regard, we succeed in determining an explicit evaluation for the partial sums of

$$\left(F_{\left\lfloor \frac{n}{s} \right\rfloor} : n \in \mathbb{N}_0\right) \tag{1.4}$$

for $s \in \mathbb{N}$. We also demonstrate how our method may be applied to evaluate the partial sums of

$$\left(F_{\left\lfloor \frac{\sqrt[r]{n}}{s} \right\rfloor} : n \in \mathbb{N}_0\right)$$

for integers $r \geq 2$ and $s \geq 1$. The problem of evaluating finite sums of expressions of the form $F_{\left\lfloor \log_2\left(\frac{n}{s}\right) \right\rfloor}$ for a fixed parameter $s \in \mathbb{N}$ is also considered, but this proves to be more difficult compared to the preceding cases.

The integer sequence indicated in (1.2) is indexed in the On-line Encyclopedia of Integer Sequences (OEIS) [16] as `A115338`, but the partial sums of (1.2), starting at, say, $n = 0$, $n = 1$, or $n = 2$, are not currently indexed in the OEIS. This suggests that our identity for

$$\sum_{n=1}^{m} F_{\left\lfloor \sqrt{n} \right\rfloor}, \tag{1.5}$$

as given in Section 2 below, is new. The integer sequence (1.3) is not currently in the OEIS, and the partial sums of (1.3) are not currently in the OEIS,

Letting $s \geq 4$, sequences of the form (1.4) do not currently appear to be included in the OEIS, and the partial sums of these sequences for $s \geq 3$ do not currently appear to be in the OEIS. Computer algebra systems, including both Maple 2020 and the latest version of Mathematica in 2022, cannot evaluate the finite sums indicated in (1.5) and in

$$\sum_{n=1}^{m} F_{\lfloor \log_2(n) \rfloor} \tag{1.6}$$

and

$$\sum_{n=1}^{m} F_{\left\lfloor \frac{n}{s} \right\rfloor}, \tag{1.7}$$

even with the use of Maple commands such as `with(combinat, fibonacci):` or Mathematica commands such as `FunctionExpand`. This motivates the practical application, in terms of the development of CAS software, of the summation techniques we have applied to evaluate (1.5)–(1.7). Sums as in (1.5)–(1.7), or, more broadly, Fibonacci and Lucas sums with summands involving floor and ceiling functions, also do not appear in classic texts such as *Fibonacci & Lucas numbers, and the golden section* [21].

With regard to the finite sum in (1.5), the partial sum identity

$$\sum_{n=1}^{m} \left\lfloor \sqrt{n} \right\rfloor = (m+1) \left\lfloor \sqrt{m} \right\rfloor - \frac{\left(\left\lfloor \sqrt{m} \right\rfloor + 1\right)^3 - \frac{3}{2}\left(\left\lfloor \sqrt{m} \right\rfloor + 1\right)^2 + \frac{\left\lfloor \sqrt{m} \right\rfloor + 1}{2}}{3} \tag{1.8}$$

is well known and included in a number of classic textbooks, namely, *Concrete mathematics* [8, p. 87], *The art of computer programming* [11, §1.2.4], and *Discrete mathematics and its application* [19, §2.4]. This classic identity has inspired us to devise methods for evaluating sums that are similar in appearance to the left-hand side of (1.8) and that involve Fibonacci numbers. This has led us to discover a remarkable identity, as given in Section 2, for sums of the form shown in (1.5).

We recall the Binet formula $F_n = \frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}}$, writing $\phi = \frac{1+\sqrt{5}}{2}$ to denote the famous mathematical constant known as the golden ratio [13]. A source of motivation behind the evaluation of sums involving both Fibonacci numbers and the floor function, as in the sums shown in (1.5)–(1.7), is due to past research [9] on lower Wythoff sequences concerning sums of the form

$$\sum_{n=1}^{m} \lfloor n\phi \rfloor .$$

For further literature concerning analytical formulas for finite sums involving the floor or ceiling or rounding functions, we cite references such as [7, 10, 14, 15, 18, 20], noting that many classical and fundamental results in number theory, such as Legendre's formula for the p-adic valuation of factorials, concern finite sums of integer parts. Our main proof technique, which relies on the finite sum rearrangement shown in (1.10), may be applied quite broadly to such past references.

1.1. **The modified Abel lemma on summation by parts.** To prove our main identities, we make use of what we refer to as an *Abel-type* summation lemma. As in [3], we record that what is known as *Abel's lemma on summation by parts* was formulated in 1826 by Niels Henrik Abel [1], and that the *modified Abel lemma on summation by parts* (cf. [4, 6, 22, 23]) may be formulated as below:

$$\sum_{n=1}^{\infty} B_n \nabla A_n = \left(\lim_{m\to\infty} A_m B_{m+1} \right) - A_0 B_1 + \sum_{n=1}^{\infty} A_n \Delta B_n \tag{1.9}$$

if this limit exists and if one of the two infinite series given above converges, letting the operators ∇ and Δ be such that $\nabla\tau_n = \tau_n - \tau_{n-1}$ and $\Delta\tau_n = \tau_n - \tau_{n+1}$ for a mapping $\tau\colon \mathbb{N}_0 \to \mathbb{C}$. In this article, we apply a finite sum identity corresponding to the series rearrangement identity shown in (1.9): For a natural number m, the following finite sum companion to (1.9) holds true:

$$\sum_{n=1}^{m} B_n \nabla A_n = A_m B_{m+1} - A_0 B_1 + \sum_{n=1}^{m} A_n \Delta B_n. \tag{1.10}$$

Our proofs of Theorems 2.1, 3.1, and 4.1 follow a similar format, as described as follows. To determine an evaluation for the finite sum

$$\sum_{n=1}^{m} F_{\lfloor g(n) \rfloor}$$

for a function g, we begin by setting $A_n = n$ and $B_n = F_{\lfloor g(n) \rfloor}$ in the finite sum version of the modified Abel lemma, as shown in (1.10). So, the formula in (1.10) gives us that

$$\sum_{n=1}^{m} F_{\lfloor g(n) \rfloor} = m F_{\lfloor g(m+1) \rfloor} + \sum_{n=1}^{m} n \left(F_{\lfloor g(n) \rfloor} - F_{\lfloor g(n+1) \rfloor} \right). \tag{1.11}$$

Then, using indicator functions, as defined below, we proceed to rewrite the latter sum in (1.11) so as to be expressible in terms of previously known Fibonacci sums. We write

$$\mathbf{1}_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S, \end{cases}$$

for a set S, and the functions g we input into (1.11) are such that the summand factor $F_{\lfloor g(n) \rfloor} - F_{\lfloor g(n+1) \rfloor}$ may be simplified using indicator functions.

2. On the sum $\sum_{n=1}^{m} F_{\lfloor \sqrt{n} \rfloor}$

Theorem 2.1. *The sum*

$$\sum_{n=1}^{m} F_{\lfloor \sqrt{n} \rfloor} \tag{2.1}$$

is equal to

$$\left(2\left\lfloor \sqrt{m+1} \right\rfloor - 3\right) F_{\left\lfloor \sqrt{m+1} \right\rfloor - 1} - \left(\left\lfloor \sqrt{m+1} \right\rfloor^2 - 2\left\lfloor \sqrt{m+1} \right\rfloor - m + 4\right) F_{\left\lfloor \sqrt{m+1} \right\rfloor} + 3$$

for all natural numbers m.

Proof. We set $g(n) = \sqrt{n}$ in (1.11). This gives us the following:

$$\begin{aligned}
\sum_{n=1}^{m} F_{\lfloor \sqrt{n} \rfloor} &= m\, F_{\lfloor \sqrt{m+1} \rfloor} - \sum_{n=1}^{m} n \left(F_{\lfloor \sqrt{n+1} \rfloor} - F_{\lfloor \sqrt{n} \rfloor} \right) \\
&= m\, F_{\lfloor \sqrt{m+1} \rfloor} - \sum_{n=1}^{m} n\, \mathbf{1}_{\mathbb{N}} \left(\sqrt{n+1}\right) F_{\lfloor \sqrt{n} \rfloor - 1} \\
&= m\, F_{\lfloor \sqrt{m+1} \rfloor} - \sum_{n=1}^{m} n\, \mathbf{1}_{\{i^2-1 : i \in \mathbb{N}\}} (n)\, F_{\lfloor \sqrt{n} \rfloor - 1} \\
&= m\, F_{\lfloor \sqrt{m+1} \rfloor} - \sum_{\substack{2 \leq \ell^2 \leq m+1 \\ n = \ell^2 - 1}} n\, F_{\lfloor \sqrt{n} \rfloor - 1} \\
&= m\, F_{\lfloor \sqrt{m+1} \rfloor} - \sum_{\ell=2}^{\lfloor \sqrt{m+1} \rfloor} \left(\ell^2 - 1\right) F_{\lfloor \sqrt{\ell^2-1} \rfloor - 1} \\
&= m\, F_{\lfloor \sqrt{m+1} \rfloor} - \sum_{\ell=2}^{\lfloor \sqrt{m+1} \rfloor} \left(\ell^2 - 1\right) F_{\ell-2}.
\end{aligned} \tag{2.2}$$

Applying reindexing to the sum shown in (2.2), this leads us to use known evaluations for $S(0, s)$, $S(1, s)$, and $S(2, s)$, for a parameter s, recalling the definition in (1.1). It is easily seen that

$$\sum_{n=0}^{s} F_n = F_{s+2} - 1 \tag{2.3}$$

and that

$$\sum_{n=0}^{s} nF_n = sF_{s+2} - F_{s+3} + 2, \tag{2.4}$$

and an evaluation for $S(2, s)$ in terms of Fibonacci numbers is given in [12, 17]. Explicitly,

$$\sum_{n=0}^{s} n^2 F_n = \left(s^2 - 2s + 5\right) F_s + \left(s^2 - 4s + 8\right) F_{s+1} - 8,$$

as in [12, 17]. So, by applying reindexing to (2.2), we find that

$$\sum_{n=1}^{m} F_{\lfloor \sqrt{n} \rfloor} = mF_{\lfloor \sqrt{m+1} \rfloor} - \sum_{\ell=0}^{\lfloor \sqrt{m+1} \rfloor - 2} \left(\ell^2 + 4\ell + 3\right) F_\ell,$$

so that the above identities for $S(0, s)$, $S(1, s)$, and $S(2, s)$ give us the desired result. □

Currently, the only reference provided for the OEIS entry corresponding to the summand in (2.1) is the text [24, p. 62], but the material on Fibonacci numbers in [24] is very general and does not apply in any meaningful way to Theorem 2.1 or our proof of this result.

3. On the sum $\sum_{n=1}^{m} F_{\lfloor \log_2(n) \rfloor}$

Letting $\log_2(x)$ denote the base-2 logarithm of a value x, proofs of the known analytical formula

$$\sum_{n=1}^{m} \lfloor \log_2(n) \rfloor = (m+1) \lfloor \log_2(m) \rfloor - 2 \left(2^{\lfloor \log_2(m) \rfloor} - 1 \right)$$

are much like those for (1.8), again with reference to Knuth's classic text *The art of computer programming* [11, §1.2.4]. This, together with our proof for Theorem 2.1, raises the question of how

$$\sum_{n=1}^{m} F_{\lfloor \log_2(n) \rfloor}$$

may be evaluated. This has led us to discover and prove the interesting result highlighted below as Theorem 3.1, where the sequence $(L_n : n \in \mathbb{N}_0)$ of Lucas numbers is defined as per usual.

Theorem 3.1. *The sum*

$$\sum_{n=1}^{m} F_{\lfloor \log_2(n) \rfloor}$$

is equal to

$$(m+1) F_{\lfloor \log_2(m+1) \rfloor} - \frac{4}{5} \left(2^{\lfloor \log_2(m+1) \rfloor - 1} L_{\lfloor \log_2(m+1) \rfloor - 1} - 2 \right) - 2$$

for all natural numbers m.

Proof. We set $g(n) = \log_2(n)$ in (1.11). This gives us that

$$\begin{aligned}
\sum_{n=1}^{m} F_{\lfloor \log_2(n) \rfloor} &= m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{n=2}^{m} n \, \mathbf{1}_{\mathbb{N}}(\log_2(n+1)) F_{\lfloor \log_2(n) \rfloor - 1} \\
&= m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{n=2}^{m} n \, \mathbf{1}_{\{2^i - 1 : i \in \mathbb{N}, i \geq 2\}}(n) F_{\lfloor \log_2(n) \rfloor - 1} \\
&= m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{\substack{2 \leq n \leq m \\ n = 2^{\ell} - 1}} n \, F_{\lfloor \log_2(n) \rfloor - 1} \\
&= m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{3 \leq 2^{\ell} \leq m+1} (2^{\ell} - 1) F_{\left\lfloor \log_2(2^{\ell} - 1) \right\rfloor - 1} \\
&= m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{\ell=2}^{\lfloor \log_2(m+1) \rfloor} \left(2^{\ell} - 1 \right) F_{\left\lfloor \log_2\left(2^{\ell} - 1\right) \right\rfloor - 1} \\
&= m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{\ell=2}^{\lfloor \log_2(m+1) \rfloor} \left(2^{\ell} - 1 \right) F_{\ell - 2}. \qquad (3.1)
\end{aligned}$$

The sequence

$$\left(\sum_{i=0}^{j} 2^i F_i : i \in \mathbb{N}_0 \right)$$

agrees with the OEIS entry `A014334` given by the exponential convolution of the Fibonacci sequence with itself [16], and it is known that

$$\sum_{i=0}^{j} 2^i F_i = \frac{1}{5} \left(2^{j+1} L_{j+1} - 2 \right) \tag{3.2}$$

for all $j \in \mathbb{N}_0$. So, by applying a reindexing argument to the sum shown in (3.1), we find that

$$\sum_{n=1}^{m} F_{\lfloor \log_2(n) \rfloor} = m F_{\lfloor \log_2(m+1) \rfloor} - 1 - \sum_{\ell=0}^{\lfloor \log_2(m+1) \rfloor - 2} \left(2^{\ell+2} - 1 \right) F_\ell.$$

So, by expanding the summand of this latter sum, we may apply the identity in (3.2) together with the Fibonacci sum identity shown in (2.3). □

It is not obvious how the above proof may be generalized so as to be applicable to sums of the form $\sum_{n=1}^{m} F_{\lfloor \log_s(n) \rfloor}$ for a natural number parameter $s > 2$, since it is not obvious how sums of the form $\sum_{n=1}^{m} s^n F_n$ may be expressed explicitly in terms of Fibonacci/Lucas numbers for $s > 2$ (see the OEIS sequences `A082987` and `A082988` for example).

4. Infinite families

Mimicking our proof of Theorem 2.1, we may prove a known formula for sums of the form $\sum_{n=1}^{m} \lfloor \sqrt[r]{n} \rfloor$ using *Faulhaber's formula* [8], which refers to the following classical identity:

$$\sum_{n=1}^{m} n^t = \frac{1}{t+1} \sum_{n=1}^{t+1} (-1)^{\delta_{n,t}} \binom{t+1}{n} B_{t+1-n} m^i,$$

writing δ to denote the Kronecker delta symbol, and letting B_i denote the i^{th} Bernoulli number. We may similarly generalize our identity highlighted as Theorem 2.1, i.e., so as to evaluate sums of the form

$$\sum_{n=1}^{m} F_{\lfloor \sqrt[r]{n} \rfloor}. \tag{4.1}$$

Explicitly, setting $g(n) = \sqrt[r]{n}$ in (1.11), we find that the following equalities hold for $r \geq 2$:

$$\begin{aligned}
\sum_{n=1}^{m} F_{\lfloor \sqrt[r]{n} \rfloor} &= m F_{\lfloor \sqrt[r]{m+1} \rfloor} + \sum_{n=1}^{m} n \left(F_{\lfloor \sqrt[r]{n} \rfloor} - F_{\lfloor \sqrt[r]{n+1} \rfloor} \right) \\
&= m F_{\lfloor \sqrt[r]{m+1} \rfloor} - \sum_{n=2^r-1}^{m} n \mathbf{1}_{\mathbb{N}}(\sqrt[r]{n+1}) F_{\lfloor \sqrt[r]{n+1} \rfloor - 2} \\
&= m F_{\lfloor \sqrt[r]{m+1} \rfloor} - \sum_{n=2^r-1}^{m} n \mathbf{1}_{\{i^r-1 : i \in \mathbb{N}\}}(n) F_{\lfloor \sqrt[r]{n+1} \rfloor - 2} \\
&= m F_{\lfloor \sqrt[r]{m+1} \rfloor} - \sum_{\ell=2}^{\lfloor \sqrt[r]{m+1} \rfloor} \left(\ell^r - 1 \right) F_{\ell-2}
\end{aligned}$$

$$= mF_{\lfloor \sqrt[r]{m+1} \rfloor} - \sum_{\ell=0}^{\lfloor \sqrt[r]{m+1} \rfloor - 1} ((\ell+2)^r - 1)\, F_\ell.$$

So, by expanding the summand factor $((\ell+2)^r - 1)$ according to the binomial theorem, and by then using known identities for evaluating sums as in (1.1) in terms of Fibonacci numbers, this gives us a way of providing a closed form for (4.1), for a given natural number $r \geq 2$.

We may similarly mimic our proofs of Theorems 2.1 and 3.1 to evaluate the partial sums of expressions of the form $F_{\lfloor \frac{n}{s} \rfloor}$, for a parameter $s \in \mathbb{N}$. The cases for $s = 1$ and $s = 2$ are easily dealt with and previously known, so we omit these base cases. Of course, if $s > m$, then the sum in (4.2) vanishes. So, we restrict our attention to the cases where $3 \leq s \leq m$.

Theorem 4.1. *For $3 \leq s \leq m$, the sum*

$$\sum_{n=1}^{m} F_{\lfloor \frac{n}{s} \rfloor} \tag{4.2}$$

is equal to

$$s\left(F_{\lfloor \frac{m+1}{s} \rfloor + 1} - 1\right) + \left(m + 1 - s\left\lfloor \frac{m+1}{s} \right\rfloor\right) F_{\lfloor \frac{m+1}{s} \rfloor}.$$

Proof. In (1.11), we set $g(n) = \frac{n}{s}$ for a parameter $s \in \mathbb{N}$. This leads us to the following:

$$\begin{aligned}
\sum_{n=1}^{m} F_{\lfloor \frac{n}{s} \rfloor} &= mF_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - \sum_{n=s}^{m} n\, \mathbf{1}_{\mathbb{N}}\left(\frac{n+1}{s}\right) F_{\lfloor \frac{n}{s} \rfloor - 1} \\
&= mF_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - \sum_{n=s}^{m} n\, \mathbf{1}_{\{si-1 : i \in \mathbb{N}\}}(n)\, F_{\lfloor \frac{n}{s} \rfloor - 1} \\
&= mF_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - \sum_{\substack{s \leq n \leq m \\ n = s\ell - 1}} n\, F_{\lfloor \frac{n}{s} \rfloor - 1} \\
&= mF_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - \sum_{\substack{2 \leq \ell \leq \frac{m+1}{s} \\ n = s\ell - 1}} n\, F_{\lfloor \frac{n}{s} \rfloor - 1} \\
&= mF_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - \sum_{\ell=2}^{\lfloor \frac{m+1}{s} \rfloor} (s\ell - 1)\, F_{\lfloor \frac{s\ell - 1}{s} \rfloor - 1} \\
&= m\, F_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - \sum_{\ell=0}^{\lfloor \frac{m+1}{s} \rfloor - 2} (s(\ell+2) - 1) F_\ell.
\end{aligned}$$

So, this gives us that the sum in (4.2) may be written as

$$mF_{\lfloor \frac{m+1}{s} \rfloor} + 1 - s - (2s - 1) \sum_{\ell=0}^{\lfloor \frac{m+1}{s} \rfloor - 2} F_\ell - s \sum_{\ell=0}^{\lfloor \frac{m+1}{s} \rfloor - 2} \ell F_\ell,$$

so that the desired result then follows directly from the Fibonacci sum identities given in (2.3) and (2.4). $\square$

We may mimic our proofs of Theorems 2.1 and 4.1 so as to evaluate summations of the following form:

$$\sum_{n=1}^{m} F_{\left\lfloor \frac{\sqrt[r]{n}}{s} \right\rfloor}. \tag{4.3}$$

For $s = 1$ or $r = 1$, the evaluation of the sum in (4.3) has been considered previously in our article. So, we let $r \geq 2$ and $s \geq 2$. Under these assumptions, by setting $g(n) = \frac{\sqrt[r]{n}}{s}$ in (1.11), we can show that (4.3) may be written as

$$mF_{\left\lfloor \frac{\sqrt[r]{m+1}}{s} \right\rfloor} + 1 - s^r - \sum_{\ell=0}^{\left\lfloor \frac{\sqrt[r]{m+1}}{s} \right\rfloor - 2} \left(((2+\ell)s)^r - 1 \right) F_\ell,$$

so that, for a given value r, by expanding the above summand, we may then apply known techniques due to Ollerton and Shannon et al. [17] (cf. [2, 12]) on the evaluation of sums of the form $S(t, m)$, as defined in (1.1).

In (1.11), we set $g(n) = \log_2 \left(\frac{n}{s} \right)$ for a natural number s. This gives us that

$$\sum_{n=1}^{m} F_{\left\lfloor \log_2\left(\frac{n}{s}\right) \right\rfloor} = mF_{\left\lfloor \log_2\left(\frac{m+1}{s}\right) \right\rfloor} + \sum_{n=1}^{m} n \left(F_{\left\lfloor \log_2\left(\frac{n}{s}\right) \right\rfloor} - F_{\left\lfloor \log_2\left(\frac{n+1}{s}\right) \right\rfloor} \right). \tag{4.4}$$

In this case, determining an identity for the second summand factor in on the right-hand side of (1.11) is more complicated, compared to our previous proofs, as we need to consider the two separate cases indicated in (4.5), adopting the usual convention for extending the Fibonacci sequence $(F_i : i \in \mathbb{N}_0)$ so as to allow negative indices for F_i, with the recursion $F_i = F_{i-1} + F_{i-2}$ holding true for all integers i.

For an integer $n \geq 1$, the difference $F_{\left\lfloor \log_2\left(\frac{n}{s}\right) \right\rfloor} - F_{\left\lfloor \log_2\left(\frac{n+1}{s}\right) \right\rfloor}$ is equal to

$$\begin{cases} -\mathbf{1}_{\{s\,2^i - 1 : i \in \mathbb{N}\}}(n) F_{\left\lfloor \log_2\left(\frac{n+1}{s}\right) \right\rfloor - 2} & \text{if } n \geq s, \\ -\mathbf{1}_{\left\{\lceil s\,2^i - 1 \rceil : i \in \left\{ \left\lceil \log_2\left(\frac{1}{s}\right) \right\rceil, \left\lceil \log_2\left(\frac{1}{s}\right) \right\rceil + 1, \ldots, 0 \right\}\right\}}(n) F_{\left\lfloor \log_2\left(\frac{n+1}{s}\right) \right\rfloor - 2} & \text{if } n < s, \end{cases} \tag{4.5}$$

letting $\lceil \cdot \rceil$ denote the ceiling function. So, for both $m < s$ and $m \geq s$, we may write the right-hand side of (4.4) as

$$mF_{\left\lfloor \log_2\left(\frac{m+1}{s}\right) \right\rfloor} - \sum_{\ell = \left\lceil \log_2\left(\frac{1}{s}\right) \right\rceil}^{\left\lfloor \log_2\left(\frac{m+1}{s}\right) \right\rfloor} \left\lceil s2^\ell - 1 \right\rceil F_{\left\lfloor \log_2\left(\frac{\lceil s2^\ell - 1 \rceil + 1}{s} \right) \right\rfloor - 2},$$

noting that $\lceil s2^\ell - 1 \rceil = s2^\ell - 1$ when $\ell \geq 0$. Although the indices of the Fibonacci numbers involved in the above summands may be easily simplified, evaluating the required sums proves to be considerably difficult, and this may serve as a suitable subject for a follow-up to our article.

5. Conclusion

All of the above proofs involve the application of the identity in (1.11). However, we may mimic (1.11) and the above proofs using sequences defined with Fibonacci-type recurrence relations, such as the Lucas sequence and the rows of Wythoff arrays; see `A035513` in the OEIS [16]. However, such generalizations would typically require appropriate analogues of the closed forms for the sums of the form $S(t, m)$, recalling (1.1). Furthermore, the input sequence $g(n)$ for (1.11) is supposed to be such that it has an inverse $h(n)$ such that $g(h(n)) = h(g(n)) = n$

and such that $h(n)$ is integer-valued. For example, this applies to $g(n) = \sqrt{n}$ and its inverse $h(n) = n^2$. Also, in all of the above proofs, we used indicator functions, via the property that

$$\lfloor g(n+1) \rfloor - \lfloor g(n) \rfloor \in \{0, 1\} \tag{5.1}$$

for all n.

From the foregoing considerations, we encourage the development of our methods using:

(1) Higher-order recurrence relations, apart from the recurrences for the Fibonacci and Lucas sequences; and
(2) Input sequences $g(n)$ that are steeper relative to the constraint indicated in (5.1).

We have considered the sequence of triangular numbers in relation to our proof of Theorem 2.1 as they form an integer sequence given by a quadratic polynomial of basic importance in many areas in mathematics. This has led us to discover an analytical formula, which appears to be new, for the OEIS sequence `A006463`. For the sake of brevity, we leave it to a future project to explore applications and generalizations of this result.

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