$\binom{pn}{n}/((p-1)n+1) \bmod p$

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Theorem 1 *If p is prime, $\binom{pn}{n}/((p-1)n+1) \equiv 1 \bmod p$ if $n = (p^k-1)/(p-1)$ for some k, and $\equiv 0 \bmod p$ otherwise.*

Proof After checking the case $n = 0$, consider the case $n \not\equiv 1 \bmod p$. Then $(p-1)n+1$ is invertible mod p, and it suffices to show that $\binom{pn}{n} \equiv 0 \bmod p$. Recall Kummer's theorem[1] on binomial coefficients: the p-adic order of $\binom{m}{n}$ is the number of carries when adding n to $m-n$ in base p. If d_j is the first nonzero base-p digit of n, the corresponding digit of pn is 0, so there is a carry in this position of the addition, and thus $\binom{pn}{n} \equiv 0 \bmod p$.

Now consider $n \equiv 1 \bmod p$. Since $(p-1)n+1 \equiv 0 \bmod p$, we use

$$\binom{pn}{n}/((p-1)n+1) = \frac{(pn)!}{(p-1)n+1)!\; n!} = \binom{pn}{n-1}/n \equiv \binom{pn}{n-1} \bmod p$$

Let the base p representation of n be $d_k d_{k-1} \ldots d_1 1$, where $d_k \neq 0$. The representation of pn is then $d_k d_{k-1} \ldots d_1 10$, and that of $n-1$ is $d_k \ldots d_1 0$. By Lucas's theorem[2],

$$\binom{5n}{n-1} \equiv \binom{1}{d_1}\binom{d_1}{d_2} \cdots \binom{d_{k-1}}{d_k}\binom{d_k}{0} \bmod p$$

In order for none of these factors to be 0, we need $1 \geq d_1 \geq d_2 \geq \ldots \geq d_k \geq 0$, i.e. all $d_j = 1$, which makes $n = \sum_{j=0}^{k} p^j = (p^{k+1}-1)/(p-1)$, and in that case all the factors are 1.

References

[1] Wikipedia, "Kumer's theorem", https://en.wikipedia.org/wiki/Kummer's_theorem

[2] Wikipedia, "Lucas's theorem", https://en.wikipedia.org/wiki/Lucas's_theorem