OEIS A111967

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Abstract. We proof a conjecture of Paul Barry of 2005 concerning the row sums of the triangular matrix in A111967.

1. Lower Triangular Matrices

Sequence [1, A101688] defines an infinite lower triangular integer matrix which contains c consecutive 1's in column c starting at the diagonal:

$$A \equiv \begin{pmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ 0 & 1 & 1 &&&&& \\ 0 & 0 & 1 & 1 &&&& \\ 0 & 0 & 1 & 1 & 1 &&& \\ 0 & 0 & 0 & 1 & 1 & 1 && \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \end{pmatrix}. \tag{1}$$

Definition 1. *A Lower Triangular Matrix (LTM) is a matrix in which all entries right from the diagonal are zero:*

$$A_{j,k} = 0, \quad k > j. \tag{2}$$

These zeros in the upper-right triangular part of the matrix are not displayed.

The inverse matrix A^{-1} of a $n \times n$ matrix is defined by requiring orthogonality between the rows and columns of A^{-1} and A:

$$\sum_{k=1}^{n} (A^{-1})_{i,k} A_{k,j} = \delta_{i,j}. \tag{3}$$

If A is a LTM, the summation over k effectively covers only the range $k \geq j$; in particular it reduces to a single term for the diagonal element:

$$(A^{-1})_{i,i} A_{i,i} = 1; \tag{4}$$

$$\sum_{k=j}^{n} (A^{-1})_{i,k} A_{k,j} = 0, \quad i \neq j. \tag{5}$$

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The second requirement is fulfilled for $j > i$ by defining A^{-1} also as a LTM:

$$(A^{-1})_{i,i} = \frac{1}{A_{i,i}}; \tag{6}$$

$$(A^{-1})_{i,k} = 0, \quad k > i, \quad j > i. \tag{7}$$

$$\sum_{k=j}^{n} (A^{-1})_{i,k} A_{k,j} = 0, \quad i > j. \tag{8}$$

The LTM proposition restricts the range of k in the previous equation to

$$\sum_{k=j}^{i} (A^{-1})_{i,k} A_{k,j} = 0, \quad i > j, \tag{9}$$

The entries in row i of A^{-1} are recursively computed right-to-left along decreasing column indices anchored at (6) via

$$(A^{-1})_{i,j} A_{j,j} + \sum_{k=j+1}^{i} (A^{-1})_{i,k} A_{k,j} = 0, \quad i > j. \tag{10}$$

$$(A^{-1})_{i,j} = -\frac{1}{A_{j,j}} \sum_{k=j+1}^{i} (A^{-1})_{i,k} A_{k,j}, \quad i > j. \tag{11}$$

The equation demonstrates: if the diagonal elements of A are all 1 and all entries of A are integer numbers, all entries of A^{-1} are also integer numbers.

2. The pair of A101688 and A111967

2.1. **Partial Sums and Recurrence.** For the specific unimodular A defined in (1), (11) simplifies to

$$(A^{-1})_{i,i} = 1; \tag{12}$$

$$(A^{-1})_{i,j} = - \sum_{k=j+1}^{i} (A^{-1})_{i,k} A_{k,j}, \quad i > j, \tag{13}$$

and because the $A_{k,j}$ are 0 for $k \geq 2j$ (using 1-based matrix indices),

$$(A^{-1})_{i,i} = 1; \tag{14}$$

$$(A^{-1})_{i,j} = - \sum_{k=j+1}^{\min(i,2j-1)} (A^{-1})_{i,k}, \quad i > j. \tag{15}$$

The entries of row i, column j of the inverse matrix are negated partial sums of entries at the same row further to the right of the inverse matrix. The upper limit i in the sum just rephrases the LTM property (7). The upper limit $2j-1$ means the summation may not even reach out to the diagonal for small column indices j. Another obvious consequence is

$$(A^{-1})_{i,1} = 0, \quad i > 1, \tag{16}$$

because the lower limit of k is larger than the upper limit. An explicit numerical evaluation gives

$$
(17)\qquad A^{-1} \equiv \left(\begin{array}{cccccccccccc}
1 & & & & & & & & & & & \\
0\,| & 1 & & & & & & & & & & \\
0\,| & -1 & 1 & & & & & & & & & \\
0 & 1\,| & -1 & 1 & & & & & & & & \\
0 & 0\,| & 0 & -1 & 1 & & & & & & & \\
0 & -1 & 1\,| & 0 & -1 & 1 & & & & & & \\
0 & 0 & 0\,| & 0 & 0 & -1 & 1 & & & & & \\
0 & 1 & -1 & 1\,| & 0 & 0 & -1 & 1 & & & & \\
0 & 0 & 0 & 0\,| & 0 & 0 & 0 & -1 & 1 & & & \\
0 & 0 & 0 & -1 & 1\,| & 0 & 0 & 0 & -1 & 1 & & \\
0 & 0 & 0 & 0 & 0\,| & 0 & 0 & 0 & 0 & -1 & 1 & \\
0 & -1 & 1 & 0 & -1 & 1\,| & 0 & 0 & 0 & 0 & -1 & 1 \\
\cdots & & & & & & & & & & &
\end{array}\right).
$$

We added vertical bars at the columns where the summation restriction $k \le 2j-1$ plays a role (i.e., for the columns left from the bar $2j-1 < i$).

2.2. **Right Halves of Lower Left Triangle.** For $i \ge 3$, (15) shows that the entries of the first sub-diagonal are the negated entries of the diagonal, $(A^{-1})_{i,i-1} = -1, i \ge 3$. The locations of the bars also indicates that the rest of the larger right half of the rows is zero, because the partial sum extends to the diagonal and therefore equals the negated $(A^{-1})_{i,i-1} + (A^{-1})_{i,i} = 0$:

$$
(18)\qquad (A^{-1})_{i,j} = 0, \quad i \ge 3, \quad (i+1)/2 \le j < i-1.
$$

2.3. **Rows with odd index.** For the entries $(A^{-1})_{i,j}$ with odd i and the "left" halves of the rows, $1 \le j < (i+1)/2$, the summation constraint $k < 2j$ plays a role. The column immediately left from the bar is $j = (i-1)/2$, where the summation is restricted to $k < i-1$. As seen in Section 2.2, the associated terms of the partial sum are all zero then, which yields $(A^{-1})_{i,(i-1)/2} = 0$. With (15), the entries more to the left are partial sums of an even smaller subset of values and all evaluate recursively to zero. So

$$
(19)\qquad (A^{-1})_{i,j} = 0, \quad \text{odd}\, i, \quad 1 \le j < i-1.
$$

Theorem 1. *This means for odd row indices i the row sums of A^{-1} are zero.*

2.4. **Rows with even index.** For the entries $(A^{-1})_{i,j}$ with even i and the "left" halves of the rows, $1 \le j < (i+1)/2$, the summation constraint $k < 2j$ plays a role. The column immediately left from the bar is $j = i/2$, where the summation is restricted to $k < i$. So the summation picks up the zeros indicated in Section 2.2 plus the negated value of $(A^{-1})_{i,i-1} = -1$, so $(A^{-1})_{i,i/2} = 1$ just left from the bar.[1] One column further to the left, the summation drops the -1 of the first sub-diagonal but grabs the $+1$ adjacent to it, so $(A^{-1})_{i,i/2-1} = -1$.[2]

So far we have established for even i that the "right" half of the row with $i/2$ entries ends with $-1, 1$, and the "left" half of the row with $i/2$ also ends with $-1, 1$ with exceptions at $i = 2$ and $i = 4$. A systematic interpretation (for even i) of (15)

[1] In fact the constraint $k \ge j+1$ leads to a different result at $i = 2, j = 1$.

[2] In fact the constraint $k \ge j+1$ leads to a different result at $i = 4, j = 1$, because the value adjacent to the left cannot be included.

is that the right half of the row except the last entry is copied sign-reversed to the left half (appearing as $+1$ left to the bar), and the value -1 one place more to the left is the sign-reversed contribution of that $+1$. These pairs of adjacent $-1, 1$ are inert for the partial sums of entries further to the left except when the effect of the upper limit $2j-1$ again grabs only the -1 of such a pair but not the $+1$. This is the reason why the evaluation next zooms into the first and second quarter of the first half, then into the first eight and second eight of the first quarter and so on, until this splitting of the initial portions breaks up because either the subdivision faces an odd number of initial values or because it reaches (16).

- If the subdivision leaves an odd number of initial values, the evaluation of Section 2.3 remains valid. The row contains an initial sequence of zeros and pairs of $-1, +1$.
- Whenever the -1 cannot be placed caused by (16), the row sum of A^{-1} is one, otherwise it is zero. These "blocked" -1 occur where $(A^{-1})_{i,2} = 1$, i.e., where via the $2j-1$ argument $(A^{-1})_{i,3} = -1$, so $(A^{-1})_{i,4} = 1$, i.e., where via the $2j-1$ argument $(A^{-1})_{i,7} = -1$, so $(A^{-1})_{i,8} = 1$ and so forth until the column on the diagonal of A^{-1} is reached. With this doubling-of-index recurrence, the blocked -1 occur whenever the column index is a power of 2.

Theorem 2. *The row sum of A^{-1} is one if the row index is a power of 2, otherwise it is zero.*

Taking into account that the OEIS uses 0-based indices of the matrices and in [1, A036987], this is exactly the statement of the conjecture.

References

1. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2021), https://oeis.org/. MR 3822822

URL: `http://www.mpia.de/~mathar`

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