Notes for A108227

For a finite **multiset** B of positive integers, define $\mathcal{P}(B) = \prod_{x\in B}\left(1+\frac{1}{x}\right)$. Note that for primes p, we have $\sigma_{-1}(p^e) = \frac{\sigma(p^e)}{p^e} = \left(1+\frac{1}{p}\right)\left(1+\frac{1}{p^2+p}\right)\left(1+\frac{1}{p^3+p^2+p}\right)\cdots\left(1+\frac{1}{p^e+p^{e-1}+\cdots+p}\right) = \mathcal{P}(\{p^j+p^{j-1}+\cdots+p : j=1,2,\cdots,e\})$. As a result, for $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$, we have $\sigma_{-1}(n) = \sigma_{-1}(p_1^{e_1})\sigma_{-1}(p_2^{e_2})\cdots\sigma_{-1}(p_r^{e_r}) = \mathcal{P}(\{p_i^j+p_i^{j-1}+\cdots+p_i : i = 1,2,\cdots,r, j=1,2,\cdots,e_i\})$.

For $k\in\mathbb{N}^*$, let $p(k)$ be the k-th prime. Define the multiset $B_k = \{p^j+p^{j-1}+\cdots+p : p \text{ prime} \ge p(k)\}$, then sort B_k into $x_1\le x_2\le x_3\le\cdots$[1]

Proposition. Write $x_i = q_i^{j_i}+q_i^{j_i-1}+\cdots+q_i$ for $i\in\mathbb{N}^*$ (in case of multiple representations, choose an arbitrary one, but a representation cannot appear more than once), then $n := \prod_{i=1}^{N} q_i$ has $\sigma_{-1}(n) = \mathcal{P}(\{x_1,x_2,\cdots,x_N\})$.

Proof. Let $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$ be the canonical factorization, then $q_i = p_1,\cdots,p_r$ for respectively $e_1,\cdots,e_r$ times. Since $x_1\le x_2\le x_3\le x_N$ are the smallest N numbers in B_k and no two numbers are allowed to have the same representation, we have exactly

$$\{x_1,x_2,\cdots,x_N\} = \{p_i^j+p_i^{j-1}+\cdots+p_i : i=1,2,\cdots,r, j=1,2,\cdots,e_i\}.$$

□

Let N_k be the first number such that $\mathcal{P}(\{x_1,x_2,\cdots,x_{N_k}\})>2$. Then the proposition above gives us automatically a $p(k)$-rough abundant number. We have moreover

Theorem. Every $p(k)$-rough abundant number has at least N_k prime factors counted with multiplicity.

Proof. Suppose that a $p(k)$-rough abundant number n has canonical factorization $n = p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$, then

$$\mathcal{P}(\{x_1,x_2,\cdots,x_{e_1+e_2+\cdots+e_r}\}) \ge \mathcal{P}(\{p_i^j+p_i^{j-1}+\cdots+p_i : i=1,2,\cdots,r, j=1,2,\cdots,e_i\}) = \sigma_{-1}(n) > 2,$$

since each $p_i^j+p_i^{j-1}+\cdots+p_i\in B_k$ and $x_1,x_2,\cdots,x_{e_1+e_2+\cdots+e_r}$ are the smallest elements in B_k. Hence $\Omega(n) = e_1+e_2+\cdots+e_r\ge N_k$. □

Theorem. For $k\in\mathbb{N}^*$, each $x_i (i=1,2,\cdots,N_k)$ is of the form p or p^2+p.

[1] Here *à priori* a number in B_k can have multiple representations, and each number should appear as many times as its number of representations, for example

$$30 = 2^4+2^3+2^2+2+1 = 5^2+5.$$

But in fact the Goormaghtigh conjecture says that 30 is the only number to have more than 1 representations.

Proof. It suffices to show that $p^3(k)+p^2(k)+p(k) > x_{N_k}$. Suppose otherwise that $p^3(k)+p^2(k)+p(k) \le x_{N_k}$, then by definition, $x_1, x_2, \cdots, x_{N_k-1}$ must include all primes in the range $[p(k), p^3(k)]$.

We need the following lemma:

Lemma ([1], p.26, Theorem 14)**.** For $x \ge 10372$, we have

$$\left| \sum_{p \text{ prime}, p\le x} \frac{1}{p} - \ln\ln x - M \right| \le \frac{1}{10\ln^2 x} + \frac{4}{15\ln^3 x},$$

where M is the Meissel-Mertens constant.

If $k \ge 1273$, then $p(k) \ge 10391 > 10372$. Write $x_i = p_i^{j_i} + p_i^{j_i-1} + \cdots + p_i$, $i = 1, 2, \cdots, N_k$, then

$$\begin{aligned}
\ln \sigma_{-1}\Big(\prod_{i=1}^{N_k-1} p_i\Big) &\ge \sum_{q \text{ prime}, p(k)<q\le p^3(k)} \ln\Big(1+\frac{1}{q}\Big) \\
&> 0.99995 \times \sum_{q \text{ prime}, p(k)<q\le p^3(k)} \frac{1}{q} \\
&> 0.99995 \times \Big(\ln\ln p^3(k) - M - \frac{1}{10\ln^2 p^3(k+1)} - \frac{4}{15\ln^3 p^3(k)} - \ln\ln p(k+1) \\
&\qquad + M - \frac{1}{10\ln^2 p(k)} - \frac{4}{15\ln^3 p(k)}\Big) \\
&= 0.99995 \times \Big(\log 3 - \frac{1}{10\ln^2 p^3(k)} - \frac{4}{15\ln^3 p^3(k)} - \frac{1}{10\ln^2 p(k)} - \frac{4}{15\ln^3 p(k)}\Big) \\
&> 1.09690.
\end{aligned}$$

Then $\prod_{i=1}^{N_k-1} p_i$ is a $p(k)$-rough abundant number with only $N_k - 1$ prime factors counted with multiplicity, a contradiction!

The case $1 \le k \le 1272$ is proved by computing $x_1, x_2, \cdots, x_{N_k}$ directly. □

k	N_k	$\{x_1, x_2, \cdots, x_{N_k}\}$	Example of $p(k)$-rough abundant number with N_k prime factors counted with multiplicity
1	3	$\{2, 3, 5\}$	$2 \times 3 \times 5$
2	5	$\{3, 5, 7, 11, 12\}$	$3^2 \times 5 \times 7 \times 11$
3	9	$\{5, 7, 11, 13, 17, 19, 23, 29, 30\}$	$5^2 \times 7 \times 11 \times \cdots \times 29$
4	18	$\{7, 11, 13, \cdots, 71, 56\}$	$5^2 \times 7 \times 11 \times \cdots \times 29$
5	31	$\{11, 13, 17, \cdots, 139, 132\}$	$11^2 \times 13 \times 17 \times \cdots \times 139$
6	46	$\{13, 17, 19, \cdots, 229, 182\}$	$13^2 \times 17 \times 19 \times \cdots \times 229$
7	67	$\{17, 19, 23, \cdots, 359, 306\}$	$17^2 \times 19 \times 23 \times \cdots \times 359$
8	91	$\{19, 23, 29, \cdots, 509, 380\}$	$19^2 \times 23 \times 29 \times \cdots \times 509$
13	284	$\{41, 43, 47, \cdots, 1931, 1722, 1892\}$	$41^2 \times 43^2 \times 47 \times 53 \times \cdots \times 1931$
14	334	$\{43, 47, 53, \cdots, 2333, 1892, 2256\}$	$43^2 \times 47^2 \times 53 \times 59 \times \cdots \times 2333$
27	1469	$\{103, 107, 109, \cdots, 12497, 10712, 11556, 11990\}$	$103^2 \times 107^2 \times 109^2 \times 113 \times 127 \times \cdots \times 12497$

Table. A demonstration for $k = 1, \cdots, 8, 13, 14, 27$

References

[1] M. B. Villarino. Mertens's proof of Merten's theorem. *arXiv:math/0504289*, Apr 2005.