Definition. Let $d(n)$ be the number of divisors of n and $\sigma(n)$ be the sum of divisors of n. If $d(n) \mid \sigma(n)$, then n is called an arithmetic number.

Lemma. (a) Let p be a prime, then every prime factor of $\dfrac{p^k-1}{p-1}$ is either a factor of k or congruent to 1 modulo k;

(b) Let p be a prime, q be an odd prime factor of k, then $v_q\Big(\dfrac{p^k-1}{p-1}\Big) \le v_q(k)$, where v_q is the $q-$adic valuation.

Theorem. Let $r \ge 1$, $p_1, p_2, \cdots, p_r$ be dintinct primes, $k_1, k_2, \cdots, k_r$ be odd numbers such that $N = p_1^{k_1-1}p_2^{k_2-1}\cdots p_r^{k_r-1}$ is an arithmetic number. Then there exists $1 \le i \le r$ such that $p_i^{k_i-1}$ is an arithmetic number.

Proof. We have $d(N) = k_1k_2\cdots k_r$ is divisble by $\sigma(N) = \dfrac{p_1^{k_1}-1}{p_1-1}\dfrac{p_2^{k_2}-1}{p_2-1}\cdots\dfrac{p_r^{k_r}-1}{p_r-1}$. For an odd prime q, define

$$I_q = \Big\{1 \le i \le r : v_q\Big(\frac{p_i^{k_i}-1}{p_i-1}\Big) < v_q(k_i)\Big\},$$

If I_q is empty for all odd primes q, then every $p_i^{k_i-1}$ is an arithmetic number.

Now suppose that q is the smallest prime such that I_q is nonempty. Since $\sum_{i=1}^{r} v_q\Big(\dfrac{p_i^{k_i}-1}{p_i-1}\Big) \ge \sum_{i=1}^{r} v_q(k_i)$, we must have $v_q\Big(\dfrac{p_i^{k_i}-1}{p_i-1}\Big) > v_q(k_i) \ge 0$ for some $1 \le i \le r$. By part (b) of the Lemma above, we have $q \not| k_i$, so by part (a) we have $q \equiv 1 (\bmod\ k_i)$, hence every prime factor of k_i is $< q$. By the minimality of q, $v_p\Big(\dfrac{p_i^{k_i}-1}{p_i-1}\Big) = v_p(k_i)$ for every prime factor p of k_i, so $p_i^{k_i-1}$ is an arithmetic number. $\square$