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5.5 Kalmár's Composition Constant

An **additive composition** of an integer n is a sequence $x_1, x_2, \ldots, x_k$ of integers (for some $k \geq 1$) such that

$$n = x_1 + x_2 + \cdots + x_k, \quad x_j \geq 1 \text{ for all } 1 \leq j \leq k.$$

A **multiplicative composition** of n is the same except

$$n = x_1 x_2 \cdots x_k, \quad x_j \geq 2 \text{ for all } 1 \leq j \leq k.$$

The number $a(n)$ of additive compositions of n is trivially 2^{n-1}. The number $m(n)$ of multiplicative compositions does not possess a closed-form expression, but asymptotically satisfies

$$\sum_{n=1}^{N} m(n) \sim \frac{-1}{\rho\zeta'(\rho)} N^{\rho} = (0.3181736521\ldots) \cdot N^{\rho},$$

where $\rho = 1.7286472389\ldots$ is the unique solution of $\zeta(x) = 2$ with $x > 1$ and $\zeta(x)$ is Riemann's zeta function [1.6]. This result was first deduced by Kalmár [1, 2] and refined in [3–8].

An **additive partition** of an integer n is a sequence $x_1, x_2, \ldots, x_k$ of integers (for some $k \geq 1$) such that

$$n = x_1 + x_2 + \cdots + x_k, \quad 1 \leq x_1 \leq x_2 \leq \cdots \leq x_k.$$

Partitions naturally represent equivalence classes of compositions under sorting. The number $A(n)$ of additive partitions of n is mentioned in [1.4.2], while the number $M(n)$ of **multiplicative partitions** asymptotically satisfies [9, 10]

$$\sum_{n=1}^{N} M(n) \sim \frac{1}{2\sqrt{\pi}} N \exp\left(2\sqrt{\ln(N)}\right) \ln(N)^{-\frac{3}{4}}.$$

Thus far we have dealt with *unrestricted* compositions and partitions. Of many possible variations, let us focus on the case in which each x_j is restricted to be a prime number. For example, the number $M_{\mathrm{p}}(n)$ of **prime multiplicative partitions** is trivially 1 for $n \geq 2$. The number $a_{\mathrm{p}}(n)$ of **prime additive compositions** is [11]

$$a_{\mathrm{p}}(n) \sim \frac{1}{\xi f'(\xi)} \left(\frac{1}{\xi}\right)^{n} = (0.3036552633\ldots) \cdot (1.4762287836\ldots)^{n},$$

where $\xi = 0.6774017761\ldots$ is the unique solution of the equation

$$f(x) = \sum_{p} x^{p} = 1, \quad x > 0,$$

and the sum is over all primes p. The number $m_{\mathrm{p}}(n)$ of **prime multiplicative compositions** satisfies [12]

$$\sum_{n=1}^{N} m_{\mathrm{p}}(n) \sim \frac{-1}{\eta g'(\eta)} N^{-\eta} = (0.4127732370\ldots) \cdot N^{-\eta},$$

where $\eta = -1.3994333287\ldots$ is the unique solution of the equation

$$g(y) = \sum_{p} p^{y} = 1, \quad y < 0.$$

Not much is known about the number $A_{\mathrm{p}}(n)$ of **prime additive partitions** [13–16] except that $A_{\mathrm{p}}(n+1) > A_{\mathrm{p}}(n)$ for $n \geq 8$.

Here is a related, somewhat artificial topic. Let p_n be the n^{th} prime, with $p_1 = 2$, and define formal series

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad Q(z) = \frac{1}{P(z)} = \sum_{n=0}^{\infty} q_n z^n.$$

Some people may be surprised to learn that the coefficients q_n obey the following asymptotics [17]:

$$q_n \sim \frac{1}{\theta P'(\theta)} \left(\frac{1}{\theta}\right)^n = (-0.6223065745\ldots) \cdot (-1.4560749485\ldots)^n.$$

where $\theta = -0.6867778344\ldots$ is the unique zero of $P(z)$ inside the disk $|z| < 3/4$. By way of contrast, $p_n \sim n\ln(n)$ by the Prime Number Theorem. In a similar spirit, consider the coefficients c_k of the $(n-1)^{\text{st}}$ degree polynomial fit

$$c_0 + c_1(x-1) + c_2(x-1)(x-2) + \cdots + c_{n-1}(x-1)(x-2)(x-3)\cdots(x-n+1)$$

to the dataset [18]

$$(1, 2), (2, 3), (3, 5), (4, 7), (5, 11), (6, 13), \ldots, (n, p_n).$$

In the limit as $n \to \infty$, the sum $\sum_{k=0}^{n-1} c_k$ converges to $3.4070691656\ldots$.

Let us return to the counting of compositions and partitions, and merely mention variations in which each x_j is restricted to be square-free [12] or where the xs must be distinct [8]. Also, compositions/partitions $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_l$ of n are said to be **independent** if proper subsequence sums/products of xs and ys never coincide. How many such pairs are there (as a function of n)? See [19] for an asymptotic answer.

Cameron & Erdös [20] pointed out that the number of sequences $1 \le z_1 < z_2 < \cdots < z_k = n$ for which $z_i | z_j$ whenever $i < j$ is $2m(n)$. The factor 2 arises because we can choose whether or not to include 1 in the sequence. What can be said about the number $c(n)$ of sequences $1 \le w_1 < w_2 < \cdots < w_k \le n$ for which $w_i \nmid w_j$ whenever $i \ne j$? It is conjectured that $\lim_{n\to\infty} c(n)^{1/n}$ exists, and it is known that $1.55967^n \le c(n) \le 1.59^n$ for sufficiently large n. For more about such sequences, known as **primitive sequences**, see [2.27].

Finally, define $h(n)$ to be the number of ways to express 1 as a sum of $n+1$ elements of the set $\{2^{-i} : i \ge 0\}$, where repetitions are allowed and order is immaterial. Flajolet & Prodinger [21] demonstrated that

$$h(n) \sim (0.2545055235\ldots)\kappa^n,$$

where $\kappa = 1.7941471875\ldots$ is the reciprocal of the smallest positive root x of the equation

$$\sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^{2^{j+1}-2-j}}{(1-x)(1-x^3)(1-x^7)\cdots(1-x^{2^j-1})} - 1 = 0.$$

This is connected to enumerating level number sequences associated with binary trees [5.6].

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5.6 Otter's Tree Enumeration Constants

A **graph** of order n consists of a set of n **vertices** (points) together with a set of **edges** (unordered pairs of distinct points). Note that loops and multiple parallel edges are automatically disallowed. Two vertices joined by an edge are called **adjacent**.

A **forest** is a graph that is **acyclic**, meaning that there is no sequence of adjacent vertices $v_0, v_1, \ldots, v_m$ such that $v_i \neq v_j$ for all $i < j < m$ and $v_0 = v_m$.

A **tree** (or **free tree**) is a forest that is **connected**, meaning that for any two distinct vertices u and w, there is a sequence of adjacent vertices $v_0, v_1, \ldots, v_m$ such that $v_0 = u$ and $v_m = w$.

Two trees σ and τ are **isomorphic** if there is a one-to-one map from the vertices of σ to the vertices of τ that preserves adjacency (see Figure 5.2). Diagrams for all non-isomorphic trees of order < 11 appear in [1]. Applications are given in [2].