Backhouse's Constant

Let P(x) be the formal power series whose nth term has coefficient equal to the nth prime number:

$$P(x) = \sum_{k=0}^{\infty} p_k \cdot x^k = 1 + 2\cdot x + 3\cdot x^2 + 5\cdot x^3 + 7\cdot x^4 + 11\cdot x^5 + 13\cdot x^6 + \ldots$$

Let Q(x) be the formal power series defined by

$$P(x)\cdot Q(x) = 1$$

Thus Q(x) is the formal reciprocal of P(x) as a power series. Observe that this is pure formal algebra: no questions of analytical convergence are involved at all.

Q(x) is an alternating series whose coefficients q_n are monotonically increasing in magnitude. Backhouse has observed that the ratios of successive coefficients tend to a certain constant, i.e., it appears that

$$\lim_{n \to \infty} \left| \frac{q_{n+1}}{q_n} \right| = 1.4560749485826896713995953511165 4356\ldots$$

In a personal communication, Backhouse wrote:

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The approximation given was generated in 37 seconds using Maple V
(release 3) in batch mode on a Silicon Graphics Irix6.  P(x) was
taken to 550 terms and Q(x) produced as the Taylor series of P(x)^(-1).

Unfortunately, I have no references to this result or anything
like it. In particular I have no evidence as to the originality
of my observation. I was just curious, as someone with an amateur
interest in number theory !

I should, of course, be very interested to hear, if, as a result
of your enterprise, someone has anything to add to my rather thin
story.
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Flajolet has proved the existence of Backhouse's constant and computed it to 1024 places, via analysis of meromorphic functions by Cauchy's coefficient formula. Here is his description of the new results.

Relevant Mathcad files will be included as time permits. Plouffe gave Flajolet's approximation in the Inverse Symbolic Calculator web pages.

Acknowledgements

I am grateful to Simon Plouffe for pointing out to me the existence of this constant, to Nigel Backhouse for providing the information on which this essay is based, and to Philippe Flajolet as always.

Postscript

Frederick Magata has found an interesting related constant. It is obtained by summing the coefficients of the n-1 degree polynomial fit to the dataset

(1,2), (2,3), (3,5), (4,7), (5,11), (6,13), ..., (n, p(n))

where p(n) is the nth prime. In the limit as n approaches infinity, the sum appears to converge to 3.4070691656... Here is more information at Plouffe's site.

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