Random Triangles

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January 21, 2010

Let X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3 be independent normally distributed random variables with mean 0 and variance 1. The points (X_1, Y_1), (X_2, Y_2), (X_3, Y_3) constitute the vertices of a triangle in Euclidean 2-space (the plane); the points (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) constitute the vertices of a triangle in Euclidean 3-space. A number of parameters (for example, sides, angles, perimeter and area) describe the triangle, but the corresponding probability density functions are not well-known. We attempt to remedy this situation in this essay. Perhaps the most famous results for random Gaussian triangles are the following [1, 2]:

$$\mathrm{P}(\text{a Gaussian triangle in 2-space is obtuse}) = 3/4 = 0.75,$$

$$\mathrm{P}(\text{a Gaussian triangle in 3-space is obtuse}) = 1 - 3\sqrt{3}/(4\pi) = 0.5865033284...$$

which translate into statements about the maximum angle exceeding $\pi/2$. Consider, however, an arbitrary angle α in a triangle. What is its first moment $\mathrm{E}(\alpha)$? This turns out to be trivial. What is its second moment $\mathrm{E}(\alpha^2)$? This is more difficult, even in 2 dimensions, and the answer is apparently new. Our essay, the first in a series, arises in an effort to expand upon [3].

0.1. Sides. Let a, b, c denote the sides of a random Gaussian triangle. The trivariate density $f(x, y, z)$ for a, b, c in 2 dimensions is [4]

$$\begin{cases} \dfrac{2}{3\pi} \dfrac{x\,y\,z}{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}} \exp\left(-\dfrac{1}{6}\left(x^2+y^2+z^2\right)\right) \\ \qquad\qquad \text{if } |x-y| < z < x+y, \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

and we shall give an elementary proof of this later. The condition $|x-y| < z < x+y$ is equivalent to $|x-z| < y < x+z$ and to $|y-z| < x < y+z$ via the Law of Cosines. As a consequence, the univariate density for a corresponds to Rayleigh's distribution:

$$\frac{x}{2} \exp\left(-\frac{x^2}{4}\right), \qquad x > 0$$

and [5, 6]

$$\mathrm{E}(a) = \sqrt{\pi} = 1.7724538509..., \qquad \mathrm{E}(a^2) = 4,$$

$$\mathrm{E}(a\,b) = 4E\left(\frac{1}{2}\right) - \frac{3}{2}K\left(\frac{1}{2}\right) = 3.3412233051....$$

where

$$K(\xi) = \int_0^{\pi/2} \frac{1}{\sqrt{1-\xi^2\sin(\theta)^2}}\,d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\xi^2 t^2)}}\,dt,$$

$$E(\xi) = \int_0^{\pi/2} \sqrt{1-\xi^2\sin(\theta)^2}\,d\theta = \int_0^1 \sqrt{\frac{1-\xi^2 t^2}{1-t^2}}\,dt$$

are complete elliptic integrals of the first and second kind [7]. The cross-correlation coefficient

$$\rho(a,b) = \frac{\mathrm{Cov}(a,b)}{\sqrt{\mathrm{Var}(a)\,\mathrm{Var}(b)}} = \frac{\mathrm{E}(a\,b)-\pi}{4-\pi} = 0.2325593465...$$

is quite small, indicating weak positive dependency. Interestingly, $\rho(a^2,b^2) = 1/4 = 0.25$ since a^2, b^2 are quadratic forms in normal variables and classical theory applies [8, 9].

The trivariate density for a, b, c in 3 dimensions is [4]

$$\begin{cases} \dfrac{\sqrt{3}}{9\pi}\,x\,y\,z\exp\left(-\dfrac{1}{6}\left(x^2+y^2+z^2\right)\right) & \text{if } |x-y|<z<x+y, \\ 0 & \text{otherwise} \end{cases}$$

which is surprisingly simpler than the corresponding result in 2 dimensions. As a consequence, the univariate density for a corresponds to the Maxwell-Boltzmann distribution:

$$\frac{x^2}{2\sqrt{\pi}}\exp\left(-\frac{x^2}{4}\right), \qquad x>0$$

and

$$\mathrm{E}(a) = \frac{4}{\sqrt{\pi}} = 2.2567583341..., \qquad \mathrm{E}(a^2) = 6,$$

$$\mathrm{E}(a\,b) = 2 + \frac{6\sqrt{3}}{\pi} = 5.3079733725...,$$

$$\rho(a,b) = \frac{-8+3\sqrt{3}+\pi}{-8+3\pi} = 0.2370510252..., \qquad \rho(a^2,b^2) = \frac{1}{4} = 0.25.$$

0.2. Perimeter and Area. For perimeter $a+b+c$, the density is a double integral:

$$\int_0^x \int_0^{x-v} f(x-u-v,u,v)\,du\,dv, \qquad x>0$$

which we have not attempted to evaluate. Thus only moments are given. In 2 dimensions,

$$\mathrm{E}(\text{perimeter}) = 3\sqrt{\pi} = 5.3173615527...,$$

$$\begin{aligned}\mathrm{E}(\text{perimeter}^2) &= \mathrm{E}((a+b+c)^2)\\ &= 3\,\mathrm{E}(a^2)+6\,\mathrm{E}(a\,b)\\ &= 12+24E\left(\frac{1}{2}\right)-9K\left(\frac{1}{2}\right) = 32.0473398308...\end{aligned}$$

and in 3 dimensions,

$$\mathrm{E}(\text{perimeter}) = \frac{12}{\sqrt{\pi}} = 6.7702750025...,$$

$$\mathrm{E}(\text{perimeter}^2) = 30+\frac{36\sqrt{3}}{\pi} = 49.8478402351....$$

More can be said about area $(1/4)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$. In 2 dimensions, area can be proved to be exponentially distributed, with density [10]

$$\frac{2}{\sqrt{3}}\exp\left(-\frac{2}{\sqrt{3}}x\right), \qquad x>0.$$

The formula given in [11] is unfortunately incorrect. In particular,

$$\mathrm{E}(\text{area}) = \frac{\sqrt{3}}{2} = 0.8660254037..., \qquad \mathrm{E}(\text{area}^2) = \frac{3}{2} = 1.5.$$

A proposed density in [12] for 3 dimensional area also seems to be wrong. We find instead

$$\mathrm{E}(\text{area}) = \sqrt{3} = 1.7320508075..., \qquad \mathrm{E}(\text{area}^2) = \frac{9}{2} = 4.5$$

and provide experimental verification elsewhere [13].

0.3. Angles. Let α, β, γ denote the angles of a random Gaussian triangle. Of course, $\alpha+\beta+\gamma=\pi$, thus γ can be eliminated from consideration. The bivariate density $\varphi(x,y)$ for α, β in 2 dimensions is [14]

$$\begin{cases} \dfrac{6}{\pi}\dfrac{\sin(x)\sin(y)\sin(x+y)}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^2} & \text{if } 0<x<\pi,\ 0<y<\pi \text{ and } x+y<\pi, \\ 0 & \text{otherwise} \end{cases}$$

and we shall confirm this later. The univariate density for α was first discovered by W. S. Kendall [15], via a fairly geometric argument, but has never appeared explicitly in the open literature (the closest was [16]; see also [17]). Starting from the bivariate density, we obtain the univariate density via

$$\begin{aligned} &\frac{6}{\pi}\int_0^{\pi-x}\frac{\sin(x)\sin(y)\sin(x+y)}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^2}dy \\ =\ & \tfrac{6}{\pi}\int_0^{\pi-x}\tfrac{\cos(x)\sin(x)}{2(4-\cos(x)^2)(\sin(x)^2+\sin(y)^2+\sin(x+y)^2)}dy \\ &+\tfrac{6}{\pi}\int_0^{\pi-x}\left(\tfrac{\sin(x)\sin(y)\sin(x+y)}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^2}-\tfrac{\cos(x)\sin(x)}{2(4-\cos(x)^2)(\sin(x)^2+\sin(y)^2+\sin(x+y)^2)}\right)dy \\ =\ & \frac{3}{\pi}\frac{\cos(x)}{\left(4-\cos(x)^2\right)^{3/2}}\left(\frac{\pi}{2}+\arcsin\left(\frac{\cos(x)}{2}\right)\right)+\frac{3}{\pi}\frac{1}{4-\cos(x)^2}. \end{aligned}$$

Call this latter expression $g(x)$. Now, since $3\,\mathrm{E}(\alpha)=\mathrm{E}(\alpha+\beta+\gamma)=\pi$, we have $\mathrm{E}(\alpha)=\pi/3$. It is harder to show that

$$\mathrm{E}(\alpha^2)=\frac{7}{36}\pi^2-\frac{1}{2}\operatorname{Li}_2\left(\frac{1}{4}\right)=1.7852634251...$$

where

$$\operatorname{Li}_2(\xi)=\sum_{k=1}^{\infty}\frac{\xi^k}{k^2}=-\int_0^{\xi}\frac{\ln(1-t)}{t}dt$$

is the dilogarithm function [18]. Also, since $3\operatorname{Var}(\alpha)+6\operatorname{Cov}(\alpha,\beta)=\operatorname{Var}(\alpha+\beta+\gamma)=0$, we have $\rho(\alpha,\beta)=-1/2$; therefore

$$\mathrm{E}(\alpha\,\beta)=\frac{5}{72}\pi^2+\frac{1}{4}\operatorname{Li}_2\left(\frac{1}{4}\right)=0.7523023542....$$

Finally,

$$G(x)=\int\limits_0^x g(\xi)\,d\xi=\frac{1}{\pi}\frac{\sin(x)}{\left(4-\cos(x)^2\right)^{1/2}}\left(\frac{\pi}{2}+\arcsin\left(\frac{\cos(x)}{2}\right)\right)+\frac{1}{\pi}x$$

which implies that $\mathrm{P}(\alpha>\pi/2)=1-G(\pi/2)=1/4=0.25$, where α is arbitrary. This is equal to $(1/3)\,\mathrm{P}(\max(\alpha,\beta,\gamma)>\pi/2)$ because a triangle can have at most one obtuse angle.

The bivariate density for α, β in 3 dimensions is new, as far as we know:

$$\begin{cases}\dfrac{24\sqrt{3}}{\pi}\dfrac{\sin(x)^2\sin(y)^2\sin(x+y)^2}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^3} & \text{if } 0<x<\pi,\ 0<y<\pi \text{ and } 0<x+y<\pi,\\ 0 & \text{otherwise.}\end{cases}$$

The univariate density for α is obtained similarly:

$$\begin{aligned}&\frac{24\sqrt{3}}{\pi}\int\limits_0^{\pi-x}\frac{\sin(x)^2\sin(y)^2\sin(x+y)^2}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^3}dy\\ &=\frac{24\sqrt{3}}{\pi}\int\limits_0^{\pi-x}\frac{\left(2+\cos(x)^2\right)\sin(x)^2}{4\left(4-\cos(x)^2\right)^2\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)}dy\\ &\quad+\frac{24\sqrt{3}}{\pi}\int\limits_0^{\pi-x}\left(\frac{\sin(x)^2\sin(y)^2\sin(x+y)^2}{\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)^3}-\frac{\left(2+\cos(x)^2\right)\sin(x)^2}{4\left(4-\cos(x)^2\right)^2\left(\sin(x)^2+\sin(y)^2+\sin(x+y)^2\right)}\right)dy\\ &=\frac{6\sqrt{3}}{\pi}\frac{\left(2+\cos(x)^2\right)\sin(x)}{\left(4-\cos(x)^2\right)^{5/2}}\left(\frac{\pi}{2}+\arcsin\left(\frac{\cos(x)}{2}\right)\right)+\frac{9\sqrt{3}}{\pi}\frac{\cos(x)\sin(x)}{\left(4-\cos(x)^2\right)^2}.\end{aligned}$$

Call this latter expression $h(x)$. We observe that $h(x)=-\sqrt{3}g'(x)$ and wonder about the meaning of such a connection. As before, $\mathrm{E}(\alpha)=\pi/3$. It follows that

$$\mathrm{E}(\alpha^2)=\frac{\pi}{3}\left(\pi-\sqrt{3}\right)=1.4760687694...,$$

$$\mathrm{E}(\alpha\,\beta)=\frac{\pi}{6}\sqrt{3}=0.9068996821....$$

Finally,

$$\mathrm{P}(\alpha>\pi/2)=1+\sqrt{3}\left(g(\pi/2)-g(0)\right)=\frac{1}{3}-\frac{\sqrt{3}}{4\pi}=0.1955011094...$$

where α is arbitrary. This again is equal to $(1/3)\,\mathrm{P}(\max(\alpha,\beta,\gamma)>\pi/2)$.

0.4. Order Statistics. We will, for brevity's sake, study only maximum/minimum angles in two dimensions and only maximum/minimum sides in three dimensions. Define $\tilde{g}(x)$ to be

$$\frac{3}{\pi}\frac{\cos(x)}{(4-\cos(x)^2)^{3/2}}\left(\frac{\pi}{2}-\arcsin\left(\frac{\cos(x)}{2}\right)-2\arctan\left(\frac{3\cos(x)}{\sqrt{4-\cos(x)^2}}\right)\right)$$
$$+\frac{3}{\pi}\frac{1-4\cos(x)^2}{(4-\cos(x)^2)(1+2\cos(x)^2)}$$

which is positive for $\pi/3 < x < \pi/2$. Given $\alpha > 0$, $\beta > 0$, $\alpha+\beta < \pi$, the angle α is maximum if $\alpha > \beta$ and $\alpha > \pi-\alpha-\beta$. Hence the density for the maximum angle is

$$\begin{cases} 3\displaystyle\int_{\pi-2x}^{x}\varphi(x,y)dy & \text{if } \pi/3 < x < \pi/2, \\ 3\displaystyle\int_{0}^{\pi-x}\varphi(x,y)dy & \text{if } \pi/2 < x < \pi \end{cases} = \begin{cases} 3\tilde{g}(x) & \text{if } \pi/3 < x < \pi/2, \\ 3g(x) & \text{if } \pi/2 < x < \pi \end{cases}$$

after breaking up the integral of $\varphi(x,y)$ precisely as outlined earlier. This density again was first discovered by Kendall [15] using a different approach. Incidentally, the identity

$$\arcsin\left(\frac{\cos(x)}{2}\right) = \arctan\left(\frac{\cos(x)}{\sqrt{4-\cos(x)^2}}\right)$$

might lead to a more natural expression for $\tilde{g}(x)$. The value $3g(\pi) = 3/\pi - 1/\sqrt{3} = 0.3775793893...$ is called the shape constant (or first collinearity constant) for planar Gaussian triangles [16, 17].

Define $\psi(x)$ to be

$$\frac{3}{\pi}\frac{\cos(x)}{(4-\cos(x)^2)^{3/2}}\left(\pi-\arcsin\left(\frac{\sqrt{4-\cos(x)^2}\sin(x)^2}{2}\right)-2\arctan\left(\frac{2+\cos(x)^2}{\cos(x)\sqrt{4-\cos(x)^2}}\right)\right)$$
$$-\frac{3}{\pi}\frac{1-4\cos(x)^2}{(4-\cos(x)^2)(1+2\cos(x)^2)}$$

which is positive for $0 < x < \pi/3$. The angle α is minimum if $\alpha < \beta$ and $\alpha < \pi-\alpha-\beta$. Hence the density for the minimum angle is

$$3\int_{x}^{\pi-2x}\varphi(x,y)dy = 3\psi(x)$$

after similar breakup. This result is evidently new. Moments for these distributions remain open.

Advancing up to three dimensions, the density for the maximum side is [4]

$$\frac{3x}{2\sqrt{\pi}}\left[2\sqrt{\frac{3}{\pi}}\left(e^{-x^2/2}-e^{-x^2/3}\right)+x\,e^{-x^2/4}\operatorname{erf}\left(\frac{\sqrt{3}x}{6}\right)\right]$$

for $x>0$, and the density for the minimum side is

$$\frac{3x}{2\sqrt{\pi}}\left[2\sqrt{\frac{3}{\pi}}\left(e^{-x^2/2}-e^{-x^2}\right)+x\,e^{-x^2/4}\operatorname{erfc}\left(\frac{\sqrt{3}x}{2}\right)\right]$$

where erf, erfc are the error and complementary error functions [19].

0.5. Trivariate Details. Our proof closely follows [20]. Consider sides a, b of a random Gaussian triangle in the plane. Using

$$a^2=(X_2-X_1)^2+(Y_2-Y_1)^2, \qquad b^2=(X_3-X_1)^2+(Y_3-Y_1)^2$$

we picture vectors $\vec{a}$, $\vec{b}$ emanating from (X_1,Y_1) to (X_2,Y_2), (X_3,Y_3), respectively. Define $0<\theta_a<2\pi$ to be the angle between vector $\vec{a}$ and the x-axis; define $0<\theta_b<2\pi$ likewise. Observe that

$$(u_a,u_b)=\left(\frac{X_2-X_1}{\sqrt{2}},\frac{X_3-X_1}{\sqrt{2}}\right), \qquad (v_a,v_b)=\left(\frac{Y_2-Y_1}{\sqrt{2}},\frac{Y_3-Y_1}{\sqrt{2}}\right)$$

are independent random vectors satisfying

$$(u_a,u_b),(v_a,v_b)\sim N\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1&\frac{1}{2}\\\frac{1}{2}&1\end{pmatrix}\right).$$

Define $s_a=a^2/4$ and $s_b=b^2/4$. Then

$$u_a=\sqrt{2s_a}\cos(\theta_a), \qquad v_a=\sqrt{2s_a}\sin(\theta_a), \qquad u_b=\sqrt{2s_b}\cos(\theta_b), \qquad v_b=\sqrt{2s_b}\sin(\theta_b)$$

and conversely

$$s_a=\frac{u_a^2+v_a^2}{2}, \qquad s_b=\frac{u_b^2+v_b^2}{2}, \qquad \tan(\theta_a)=\frac{v_a}{u_a}, \qquad \tan(\theta_b)=\frac{v_b}{u_b}.$$

The Jacobian matrix of the transformation $(u_a,v_a,u_b,v_b)\mapsto(s_a,s_b,\theta_a,\theta_b)$ is

$$J=\begin{pmatrix}u_a&v_a&0&0\\0&0&u_b&v_b\\-\frac{v_a}{u_a^2+v_a^2}&\frac{u_a}{u_a^2+v_a^2}&0&0\\0&0&-\frac{v_b}{u_b^2+v_b^2}&\frac{u_b}{u_b^2+v_b^2}\end{pmatrix}$$

For example,

$$\sec(\theta_a)^2 \frac{\partial \theta_a}{\partial u_a} = \frac{\partial}{\partial u_a} \tan(\theta_a) = \frac{\partial}{\partial u_a} \frac{v_a}{u_a} = -\frac{v_a}{u_a^2}$$

implies that

$$\frac{\partial \theta_a}{\partial u_a} = -\cos(\theta_a)^2 \frac{v_a}{u_a^2} = -\frac{u_a^2}{2s_a} \frac{v_a}{u_a^2} = -\frac{v_a}{u_a^2 + v_a^2}.$$

As another example,

$$\sec(\theta_a)^2 \frac{\partial \theta_a}{\partial v_a} = \frac{\partial}{\partial v_a} \tan(\theta_a) = \frac{\partial}{\partial v_a} \frac{v_a}{u_a} = \frac{1}{u_a}$$

implies that

$$\frac{\partial \theta_a}{\partial v_a} = \cos(\theta_a)^2 \frac{1}{u_a} = \frac{u_a^2}{2s_a} \frac{1}{u_a} = \frac{u_a}{u_a^2 + v_a^2}.$$

Since the absolute determinant $|J| = 1$, changing variables from (u_a, v_a, u_b, v_b) to $(s_a, s_b, \theta_a, \theta_b)$ is easily performed. The density of (u_a, u_b) gives rise to

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{1-(\frac{1}{2})^2}} \exp\left[-\frac{1}{2\left(1-(\frac{1}{2})^2\right)}\left(u_a^2 - 2(\tfrac{1}{2})u_a u_b + u_b^2\right)\right] \\
&= \frac{1}{\sqrt{3}\pi} \exp\left[-\frac{2}{3}\left(u_a^2 - u_a u_b + u_b^2\right)\right] \\
&= \frac{1}{\sqrt{3}\pi} \exp\left[-\frac{2}{3}\left(2s_a\cos(\theta_a)^2 - \sqrt{2s_a}\sqrt{2s_b}\cos(\theta_a)\cos(\theta_b) + 2s_b\cos(\theta_b)^2\right)\right] \\
&= \frac{1}{\sqrt{3}\pi} \exp\left[-\frac{4}{3}\left(s_a\cos(\theta_a)^2 - \sqrt{s_a s_b}\cos(\theta_a)\cos(\theta_b) + s_b\cos(\theta_b)^2\right)\right]
\end{aligned}$$

and the density of (v_a, v_b) likewise gives rise to

$$\begin{aligned}
&\frac{1}{\sqrt{3}\pi} \exp\left[-\frac{2}{3}\left(v_a^2 - v_a v_b + v_b^2\right)\right] \\
&= \frac{1}{\sqrt{3}\pi} \exp\left[-\frac{4}{3}\left(s_a\sin(\theta_a)^2 - \sqrt{s_a s_b}\sin(\theta_a)\sin(\theta_b) + s_b\sin(\theta_b)^2\right)\right].
\end{aligned}$$

By independence, the density of (u_a, u_b, v_a, v_b) is

$$\frac{1}{3\pi^2} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\theta_a - \theta_b) + s_b\right)\right]$$

where $0 < \theta_a < 2\pi$, $0 < \theta_b < 2\pi$.

We move toward integrating out θ_a. Let $\omega = \theta_a - \theta_b$. The Jacobian matrix of the transformation $(s_a, s_b, \theta_a, \theta_b) \mapsto (s_a, s_b, \omega, \theta_a)$ is

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and $|K| = 1$, hence the density of $(s_a, s_b, \omega, \theta_a)$ is

$$\frac{1}{3\pi^2} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\omega) + s_b\right)\right]$$

where $-2\pi < \omega < 2\pi$ plus an additional condition. If $\omega < 0$, then $\theta_b < 2\pi$ forces $\theta_a < 2\pi + \theta_a - \theta_b = 2\pi + \omega$, thus

$$\begin{aligned} &\frac{1}{3\pi^2} \int\limits_0^{2\pi+\omega} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\omega) + s_b\right)\right] d\theta_a \\ = \; &\frac{2\pi + \omega}{3\pi^2} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\omega) + s_b\right)\right]; \end{aligned}$$

if $\omega > 0$, then $\theta_b > 0$ forces $\theta_a > \theta_a - \theta_b = \omega$, thus

$$\begin{aligned} &\frac{1}{3\pi^2} \int\limits_\omega^{2\pi} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\omega) + s_b\right)\right] d\theta_a \\ = \; &\frac{2\pi - \omega}{3\pi^2} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\omega) + s_b\right)\right]. \end{aligned}$$

In either case, the coefficient numerator is $2\pi - |\omega|$ and the density is symmetric in ω about 0. Let $\gamma = |\omega|$, then we multiply by 2 to obtain the density of (s_a, s_b, γ) :

$$\frac{2(2\pi - \gamma)}{3\pi^2} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\gamma) + s_b\right)\right]$$

where $0 < \gamma < 2\pi$. Adding contributions at γ and $2\pi - \gamma$ yields

$$\frac{4}{3\pi} \exp\left[-\frac{4}{3}\left(s_a - \sqrt{s_a s_b}\cos(\gamma) + s_b\right)\right]$$

for $0 < \gamma < \pi$, which works since $2(2\pi - \gamma) + 2\gamma = 4\pi$ and $\cos(\gamma) = \cos(2\pi - \gamma)$. Replacing s_a, s_b by $a^2/4$, $b^2/4$ yields

$$\begin{aligned} & \frac{4}{3\pi} \exp\left[-\frac{4}{3}\left(\frac{a^2}{4} - \frac{a\,b}{4}\cos(\gamma) + \frac{b^2}{4}\right)\right] \frac{a}{2}\frac{b}{2} \\ = & \frac{1}{3\pi} a\,b \exp\left[-\frac{1}{3}\left(a^2 - a\,b\cos(\gamma) + b^2\right)\right]. \end{aligned}$$

This is already useful for computing moments of area:

$$\mathrm{E}\left(\left(\frac{1}{2} a\,b \sin(\gamma)\right)^m\right) = m! \left(\frac{\sqrt{3}}{2}\right)^m$$

for all positive integers m. Also, an initial step in calculating $\mathrm{E}(a\,b)$ is to evaluate

$$\frac{1}{3\pi}\int_0^\pi a^2 b^2 \exp\left[-\frac{1}{3}\left(a^2 - a\,b\cos(\gamma) + b^2\right)\right] d\gamma = \frac{a^2 b^2}{3} \exp\left[-\frac{1}{3}\left(a^2 + b^2\right)\right] I_0\left(\frac{a\,b}{3}\right)$$

where $I_0(z)$ is the modified Bessel function of the first kind [21]. Note that the angle γ is adjacent to sides a, b and opposite to side c, as is traditional. The analogous density for (α, β, c) appears in the next section.

We now bring c into the trivariate density, removing γ. Differentiating the Law of Cosines

$$c^2 = a^2 - 2\,a\,b\cos(\gamma) + b^2$$

with respect to γ, it is clear that

$$\begin{aligned} 2\,c\,dc &= 2\,a\,b\sin(\gamma)\,d\gamma \\ &= \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\,d\gamma \end{aligned}$$

by a formula for area, and hence the density becomes

$$\begin{aligned} & \frac{1}{3\pi}\,a\,b\exp\left[-\frac{1}{3}\left(a^2 - a\,b\cos(\gamma) + b^2\right)\right] da\,db\,d\gamma \\ = & \frac{1}{3\pi}\,a\,b\exp\left[-\frac{1}{6}\left(a^2 + b^2 + (a^2 - 2\,a\,b\cos(\gamma) + b^2)\right)\right] da\,db\,d\gamma \\ = & \frac{2}{3\pi}\frac{a\,b\,c}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \exp\left[-\frac{1}{6}\left(a^2 + b^2 + c^2\right)\right] da\,db\,dc \end{aligned}$$

assuming $0 < \gamma < \pi$, that is, $a^2 - 2\,a\,b + b^2 < c^2 < a^2 + 2\,a\,b + b^2$. The required condition $|a - b| < c < a + b$ does not change upon permutation of sides a, b, c.

Note that the variables s_a, s_b are each exponentially distributed with mean 1, with cross-correlation $1/4$. A closed-form expression for the density of (s_a, s_b) is not possible [20], but an infinite series representation [22]

$$\sum_{n=0}^{\infty} \frac{1}{4^n} \Phi(-n, 1, s_a) \Phi(-n, 1, s_b) \exp(-(s_a + s_b))$$

is valid, where $\Phi(u, v, w)$ is the confluent hypergeometric function of the first kind [23]. In this special case,

$$\Phi(-n, 1, t) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} t^k.$$

Proving the series representation makes use of

$$s_a = \left(\frac{u_a}{\sqrt{2}}\right)^2 + \left(\frac{v_a}{\sqrt{2}}\right)^2, \qquad s_b = \left(\frac{u_b}{\sqrt{2}}\right)^2 + \left(\frac{v_b}{\sqrt{2}}\right)^2$$

and the fact that $u_a/\sqrt{2}$, $u_b/\sqrt{2}$ are jointly normal with mean 0, variance $1/2$ and cross-correlation $1/2$. Other multivariate generalizations of the exponential distribution are found in [24].

For the (a, b, c)-density of random Gaussian triangles in 3-space, we refer to [4].

0.6. Bivariate Details. Let $\Delta = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$ for convenience. The transformation $(a, b, c) \mapsto (\alpha, \beta, c)$ is prescribed via

$$\cos(\alpha) = \frac{-a^2 + b^2 + c^2}{2\,b\,c}, \qquad \cos(\beta) = \frac{-b^2 + a^2 + c^2}{2\,a\,c}.$$

We have, for example,

$$-\sin(\alpha)\frac{\partial \alpha}{\partial a} = -\frac{a}{b\,c}, \qquad -\sin(\alpha)\frac{\partial \alpha}{\partial b} = \frac{a^2 + b^2 - c^2}{2\,b^2 c}, \qquad -\sin(\alpha)\frac{\partial \alpha}{\partial c} = \frac{a^2 - b^2 + c^2}{2\,b\,c^2}$$

hence

$$\frac{\partial \alpha}{\partial a} = \frac{a}{b\,c}\frac{1}{\sin(\alpha)} = \frac{a}{b\,c}\frac{1}{\sqrt{1 - \cos(\alpha)^2}} = \frac{a}{b\,c}\frac{2b\,c}{\sqrt{\Delta}} = \frac{2a}{\sqrt{\Delta}},$$

$$\frac{\partial \alpha}{\partial b} = -\frac{a^2 + b^2 - c^2}{2b^2 c}\frac{1}{\sin(\alpha)} = -\frac{a^2 + b^2 - c^2}{2b^2 c}\frac{2\,b\,c}{\sqrt{\Delta}} = -\frac{a^2 + b^2 - c^2}{b\sqrt{\Delta}},$$

$$\frac{\partial \alpha}{\partial c} = -\frac{a^2 - b^2 + c^2}{2b\,c^2}\frac{1}{\sin(\alpha)} = -\frac{a^2 - b^2 + c^2}{2b\,c^2}\frac{2\,b\,c}{\sqrt{\Delta}} = -\frac{a^2 - b^2 + c^2}{c\sqrt{\Delta}}.$$

The corresponding Jacobian matrix is

$$L = \begin{pmatrix} \dfrac{2a}{\sqrt{\Delta}} & \dfrac{-a^2-b^2+c^2}{b\sqrt{\Delta}} & \dfrac{-a^2+b^2-c^2}{c\sqrt{\Delta}} \\ \dfrac{-a^2-b^2+c^2}{a\sqrt{\Delta}} & \dfrac{2b}{\sqrt{\Delta}} & \dfrac{a^2-b^2-c^2}{c\sqrt{\Delta}} \\ 0 & 0 & 1 \end{pmatrix}$$

and $|L| = 1/(a\,b)$. By the Law of Sines,

$$a = c\frac{\sin(\alpha)}{\sin(\gamma)} = c\frac{\sin(\alpha)}{\sin(\alpha+\beta)}, \qquad b = c\frac{\sin(\beta)}{\sin(\gamma)} = c\frac{\sin(\beta)}{\sin(\alpha+\beta)}$$

and, under the change of variables,

$$\sqrt{\Delta} = 2c^2\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha+\beta)}.$$

The density of (α, β, c) in two dimensions is

$$\begin{aligned} & \frac{2}{3\pi}\frac{a^2b^2c}{\sqrt{\Delta}}\exp\left[-\frac{1}{6}\left(a^2+b^2+c^2\right)\right] \\ = \; & \frac{2c^5}{3\pi}\frac{\sin(\alpha)^2\sin(\beta)^2}{\sin(\alpha+\beta)^4\sqrt{\Delta}}\exp\left[-\frac{c^2}{6\sin(\alpha+\beta)^2}\left(\sin(\alpha)^2+\sin(\beta)^2+\sin(\alpha+\beta)^2\right)\right] \\ = \; & \frac{c^3}{3\pi}\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha+\beta)^3}\exp\left[-\frac{c^2}{6}\frac{\sin(\alpha)^2+\sin(\beta)^2+\sin(\alpha+\beta)^2}{\sin(\alpha+\beta)^2}\right]. \end{aligned}$$

Integrating out c is facilitated by observing that

$$\int_0^\infty c^3\exp\left(-\frac{c^2}{6}r\right)dc = \frac{18}{r^2}$$

for $r > 0$, therefore the density of (α, β) in two dimensions is

$$\begin{aligned} & \frac{18}{3\pi}\frac{\sin(\alpha)\sin(\beta)}{\sin(\alpha+\beta)^3}\left(\frac{\sin(\alpha+\beta)^2}{\sin(\alpha)^2+\sin(\beta)^2+\sin(\alpha+\beta)^2}\right)^2 \\ = \; & \frac{6}{\pi}\frac{\sin(\alpha)\sin(\beta)\sin(\alpha+\beta)}{(\sin(\alpha)^2+\sin(\beta)^2+\sin(\alpha+\beta)^2)^2}. \end{aligned}$$

Similarly, the density of (α, β, c) in three dimensions is

$$\begin{aligned} & \frac{\sqrt{3}}{9\pi}a^2b^2c\exp\left(-\frac{1}{6}\left(a^2+b^2+c^2\right)\right) \\ = \; & \frac{\sqrt{3}c^5}{9\pi}\frac{\sin(\alpha)^2\sin(\beta)^2}{\sin(\alpha+\beta)^4}\exp\left[-\frac{c^2}{6}\frac{\sin(\alpha)^2+\sin(\beta)^2+\sin(\alpha+\beta)^2}{\sin(\alpha+\beta)^2}\right]. \end{aligned}$$

Here we observe that

$$\int_0^\infty c^5 \exp\left(-\frac{c^2}{6}r\right) dc = \frac{216}{r^3}$$

for $r > 0$, therefore the density of (α, β) in three dimensions is

$$\begin{aligned} & \frac{216\sqrt{3}}{9\pi} \frac{\sin(\alpha)^2 \sin(\beta)^2}{\sin(\alpha+\beta)^4} \left(\frac{\sin(\alpha+\beta)^2}{\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha+\beta)^2} \right)^3 \\ = \; & \frac{24\sqrt{3}}{\pi} \frac{\sin(\alpha)^2 \sin(\beta)^2 \sin(\alpha+\beta)^2}{(\sin(\alpha)^2 + \sin(\beta)^2 + \sin(\alpha+\beta)^2)^3}. \end{aligned}$$

We turn attention to the most interesting of our moment evaluations, that concerning $\mathrm{E}(\alpha^2)$. First,

$$\int_0^\pi \arcsin\left(\frac{\cos(x)}{2}\right) dx = 0$$

because $\arcsin(\cos(\pi - x)/2) = \arcsin(-\cos(x)/2) = -\arcsin(\cos(x)/2)$. Consequently

$$\begin{aligned} \int_0^\pi \frac{x \sin(x)}{\sqrt{4 - \cos(x)^2}} dx &= -x \arcsin\left(\frac{\cos(x)}{2}\right)\Bigg|_0^\pi + \int_0^\pi \arcsin\left(\frac{\cos(x)}{2}\right) dx \\ &= \frac{\pi^2}{6} \end{aligned}$$

using integration by parts. Second,

$$\begin{aligned} \int_0^\pi \left(\arcsin\left(\frac{\cos(x)}{2}\right)\right)^2 dx &= \frac{1}{4} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{16^{m+n}} \binom{2m}{m} \binom{2n}{n} \frac{1}{2m+1} \frac{1}{2n+1} \int_0^\pi \cos(x)^{2m+2n+2} dx \\ &= \frac{\pi}{16} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{1}{64^{m+n}} \binom{2m}{m} \binom{2n}{n} \binom{2m+2n+2}{m+n+1} \frac{1}{2m+1} \frac{1}{2n+1} \\ &= \frac{\pi}{2} \operatorname{Li}_2\left(\frac{1}{4}\right) \end{aligned}$$

which is a curious generalization of sums found in [25]. Consequently

$$\begin{aligned} \int_0^\pi \frac{x \sin(x)}{\sqrt{4 - \cos(x)^2}} \arcsin\left(\frac{\cos(x)}{2}\right) dx &= -\frac{x}{2} \left(\arcsin\left(\frac{\cos(x)}{2}\right)\right)^2 \Bigg|_0^\pi + \frac{1}{2} \int_0^\pi \left(\arcsin\left(\frac{\cos(x)}{2}\right)\right)^2 dx \\ &= -\frac{\pi^3}{72} + \frac{\pi}{4} \operatorname{Li}_2\left(\frac{1}{4}\right) \end{aligned}$$

using integration by parts again. Third, $G(\pi) = 1$ and $G(0) = 0$, where $G'(x) = g(x)$. Finally,

$$\begin{aligned}
\int_0^\pi x^2 G'(x) dx &= x^2 G(x)\big|_0^\pi - 2\int_0^\pi x\, G(x)\, dx \\
&= \pi^2 - \frac{2}{\pi}\int_0^\pi x \frac{\sin(x)}{\sqrt{4-\cos(x)^2}}\left(\frac{\pi}{2} + \arcsin\left(\frac{\cos(x)}{2}\right)\right) dx - \frac{2}{\pi}\int_0^\pi x^2 dx \\
&= \pi^2 - \frac{\pi^2}{6} - 2\left(-\frac{\pi^2}{72} + \frac{1}{4}\operatorname{Li}_2\left(\frac{1}{4}\right)\right) - \frac{2}{3}\pi^2 \\
&= \frac{7}{36}\pi^2 - \frac{1}{2}\operatorname{Li}_2\left(\frac{1}{4}\right)
\end{aligned}$$

as was to be shown.

A random Gaussian triangle *captures* a location (ξ, η) with probability

$$\frac{3}{(2\pi)^{5/2}}\left[\varphi(\delta) + \psi(\delta)\right] = \begin{cases} 0.250000... & \text{if } \delta = 0, \\ 0.197171... & \text{if } \delta = 1/2, \\ 0.098289... & \text{if } \delta = 1, \\ 0.032455... & \text{if } \delta = 3/2, \\ 0.007626... & \text{if } \delta = 2 \end{cases}$$

where $\delta = \sqrt{\xi^2 + \eta^2}$ and

$$\varphi = \int_0^\infty \int_0^\infty \int_{-\infty}^0 \exp\left(-\tfrac{(a_1+\delta)^2+(b_1+\delta)^2+(c_1+\delta)^2}{2}\right)\left[\pi + 2\arctan\left(\tfrac{a_1 b_1}{c_1\sqrt{a_1^2+b_1^2+c_1^2}}\right)\right] dc_1 db_1 da_1,$$

$$\psi = \int_{-\infty}^0 \int_{-\infty}^0 \int_0^\infty \exp\left(-\tfrac{(a_1+\delta)^2+(b_1+\delta)^2+(c_1+\delta)^2}{2}\right)\left[\pi - 2\arctan\left(\tfrac{a_1 b_1}{c_1\sqrt{a_1^2+b_1^2+c_1^2}}\right)\right] dc_1 db_1 da_1.$$

The specific result $1/4$ for capturing $(0, 0)$ is well-known [26]; the general result is less so [27]. See also [28, 29, 30].

We conclude with an unsolved problem: what is an exact expression for

$$\operatorname{E}(a\,\gamma) = \frac{1}{3\pi}\int_0^\infty \int_0^\infty \int_0^\pi x^2\, y\, \theta \exp\left[-\frac{1}{3}\left(x^2 - x\, y\cos(\theta) + y^2\right)\right] d\theta\, dy\, dx = 1.6377...$$

(in two dimensions)? An answer for $\operatorname{E}(a\,\alpha)$ is believed to be even more difficult.

0.7. Acknowledgements. I am grateful to Wilfrid Kendall, Stephen Herschkorn, James Ferry and William Gosper for their kind help, and to the creators of both Mathematica and Maple.

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