On Certain Computational and Geometric Properties of Complex Horadam Orbits

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Introduction

Numerous geometric patterns identified in nature, art or science can be generated from recurrent sequences, such as for example certain fractals or Fermat's spiral. The Fibonacci numbers defined by the recurrence

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 1, F_1 = 1, \tag{1}$$

are ubiquitous in nature patterns and inspired the design of search techniques, pseudo-random number generators, or structures with optimal properties.

Horadam Sequences

The Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is a natural extension of the Fibonacci numbers to the complex plane, defined by the recurrence

$$w_{n+2} = pw_{n+1} + qw_n, \quad w_0 = a, w_1 = b, \tag{2}$$

where the parameters a, b, p and q are complex numbers. When $(a, b) = (0, 1)$, $(p, q) = (1, 1)$ gives the Fibonacci, while $(p, q) = (1, -1)$ the Lucas sequence.

Generators The roots z_1 and z_2 of the quadratic below are called generators.

$$P(x) = x^2 - px - q \tag{3}$$

General sequence term $(z_1 \neq z_2)$ The general term of the sequence $\{w_n\}_{n=0}^{\infty}$ is

$$w_n = Az_1^n + Bz_2^n. \tag{4}$$

The constants A and B are obtained from the initial values $w_0 = a, w_1 = b$.

Periodicity $z_1 = e^{2\pi i p_1/k_1} \neq z_2 = e^{2\pi i p_2/k_2}$ where p_1, p_2, k_1, k_2 are natural numbers.

Geometric bounds of periodic orbits Periodic orbits are located inside the annulus

$$\{z \in \mathbb{C} : ||A| - |B|| \leq |z| \leq |A| + |B|\}. \tag{5}$$

Properties of Periodic Horadam Orbits

Enumeration formulae $(AB \neq 0,\ z_1 \neq z_2)$ The function enumerating the number of Horadam sequences $\{w_n\}_{n=0}^{\infty}$ having period k is denoted by $H_P(k)$.

$$H_P(k) = \sharp\{(p_1, k_1, p_2, k_2) : (p_1, k_1) = (p_2, k_2) = 1,\ [k_1, k_2] = k,\ k_1 \leq k_2\},$$
$$= \sum_{[k_1,k_2]=k,\, k_1<k_2} \varphi(k_1)\varphi(k_2) + \frac{1}{2}\varphi(k)\left(\varphi(k) - 1\right), \tag{6}$$
$$= \left[\sum_{d|k,\, d<k} \varphi(d)2^{\omega(k/d)} + \varphi(k) - 1\right]\frac{\varphi(k)}{2}, \tag{7}$$

where φ is Euler's totient function and ω the number of prime divisors.

Example The following pairs $\left(\frac{p_1}{k_1}, \frac{p_2}{k_2}\right)$ produce all orbits of period $k = 6$

$$\left\{\left(\tfrac{1}{1}, \tfrac{1}{6}\right), \left(\tfrac{1}{1}, \tfrac{5}{6}\right), \left(\tfrac{1}{2}, \tfrac{1}{3}\right), \left(\tfrac{1}{2}, \tfrac{2}{3}\right), \left(\tfrac{1}{2}, \tfrac{1}{6}\right), \left(\tfrac{1}{2}, \tfrac{5}{6}\right), \left(\tfrac{1}{3}, \tfrac{1}{6}\right), \left(\tfrac{1}{3}, \tfrac{5}{6}\right), \left(\tfrac{2}{3}, \tfrac{1}{6}\right), \left(\tfrac{2}{3}, \tfrac{5}{6}\right), \left(\tfrac{1}{6}, \tfrac{5}{6}\right)\right\}$$

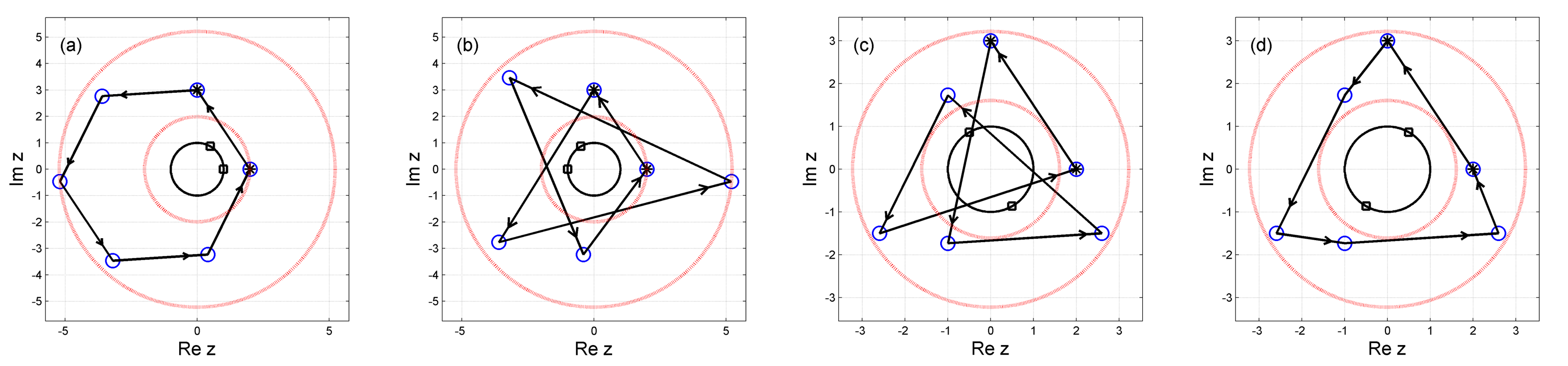


Figure 1: Sequence orbit $\{w_n\}_{n=0}^{6}$ given by $(\frac{p_1}{k_1}, \frac{p_2}{k_2})$ (a) $(\frac{1}{1}, \frac{1}{6})$; (b) $(\frac{1}{2}, \frac{1}{3})$; (c) $(\frac{1}{3}, \frac{5}{6})$; (d) $(\frac{2}{3}, \frac{1}{6})$. Also plotted: a, b (stars), z_1, z_2 (squares), unit circle (solid line), annulus (5) (dotted line).

Integer sequence $H_P(k)$ gives the context for the O.E.I.S. sequence no. A102309

$$1, 1, 3, 5, 10, 11, 21, 22, 33, 34, 55, 46, 78, 69, 92, 92, 136, 105, \ldots$$

Square-free formula Let $m \geq 2$, $p_1, \ldots, p_m$ be primes and $k = p_1 p_2 \ldots p_m$. Then

$$H_P(k) = \left[(p_1 + 1)\cdots(p_m + 1) - 1\right]\frac{(p_1 - 1)\cdots(p_m - 1)}{2} \tag{8}$$

Asymptotic bounds The following inequalities are true

$$\frac{(k-1)k}{2} \geq H_P(k) \geq \frac{\varphi(k)k}{2} \quad \left(\frac{\varphi(k)[2k - \varphi(k) - 1]}{2} \text{ if } k \text{ square-free}\right) \tag{9}$$

Geometric structure Let $k_1, k_2, d \geq 2$ be natural numbers s.t. $\gcd(k_1, k_2) = d$ and z_1, z_2 be k_1-th and k_2-th primitive roots, respectively. The orbit of $\{w_n\}_{n=0}^{\infty}$ is a k_1k_2/d-gon, representing k_1 regular k_2/d-gons or k_2 regular k_1/d-gons.

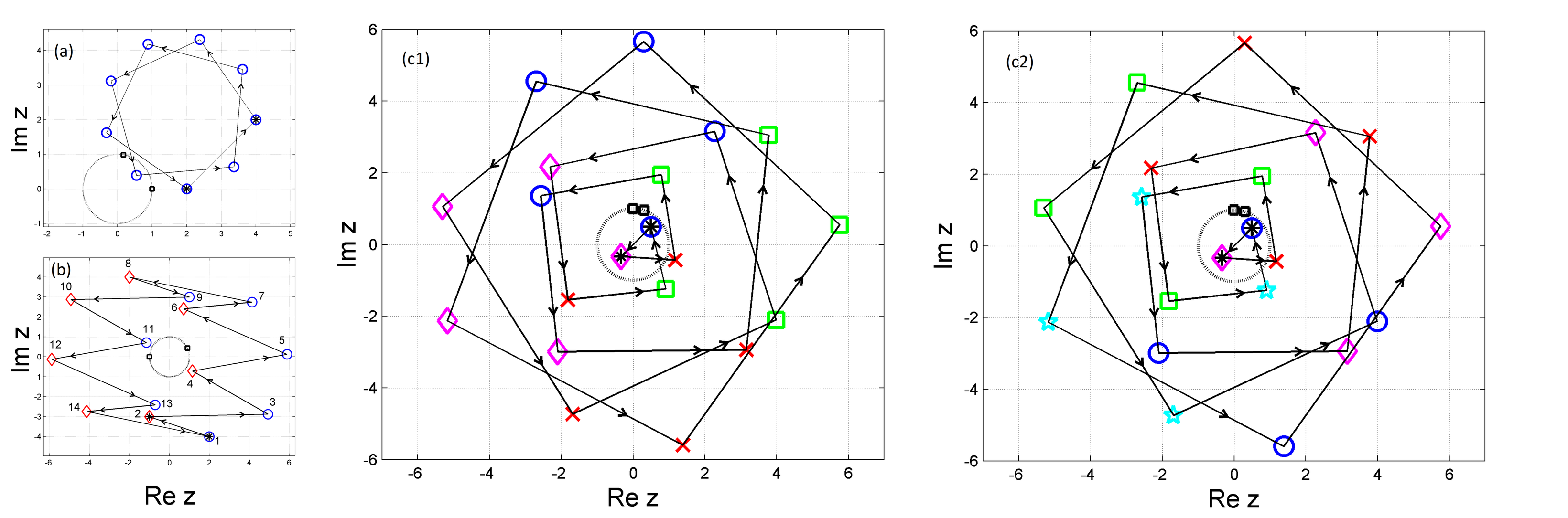


Figure 2: Periodic patterns: (a) Star Polygon; (b) Bipartite digraph; (c) Multi-symmetric. Sequence orbit $\{w_n\}_{n=0}^{20}$ obtained for $z_1 = e^{2\pi i\frac{1}{5}}$, $z_2 = e^{2\pi i\frac{1}{4}}$, $(a = (1+i)/2,\ b = -(1+i)/3)$. The orbit can be partitioned into $(c1)$ four regular pentagons; $(c2)$ five squares.

Aperiodic Horadam Orbits

Asymptotic behaviour of Horadam orbits For distinct $z_1 = r_1 e^{2\pi i x_1}$, $z_2 = r_1 e^{2\pi i x_2}$ $(r_1 \leq r_2)$, the following patterns emerge

- Stable if $r_1 = r_2 = 1$ (unless periodic);
- Quasi-convergent if $0 \leq r_1 < r_2 = 1$;
- Convergent if $0 \leq r_1 \leq r_2 < 1$;
- Divergent if $r_2 > 1$.

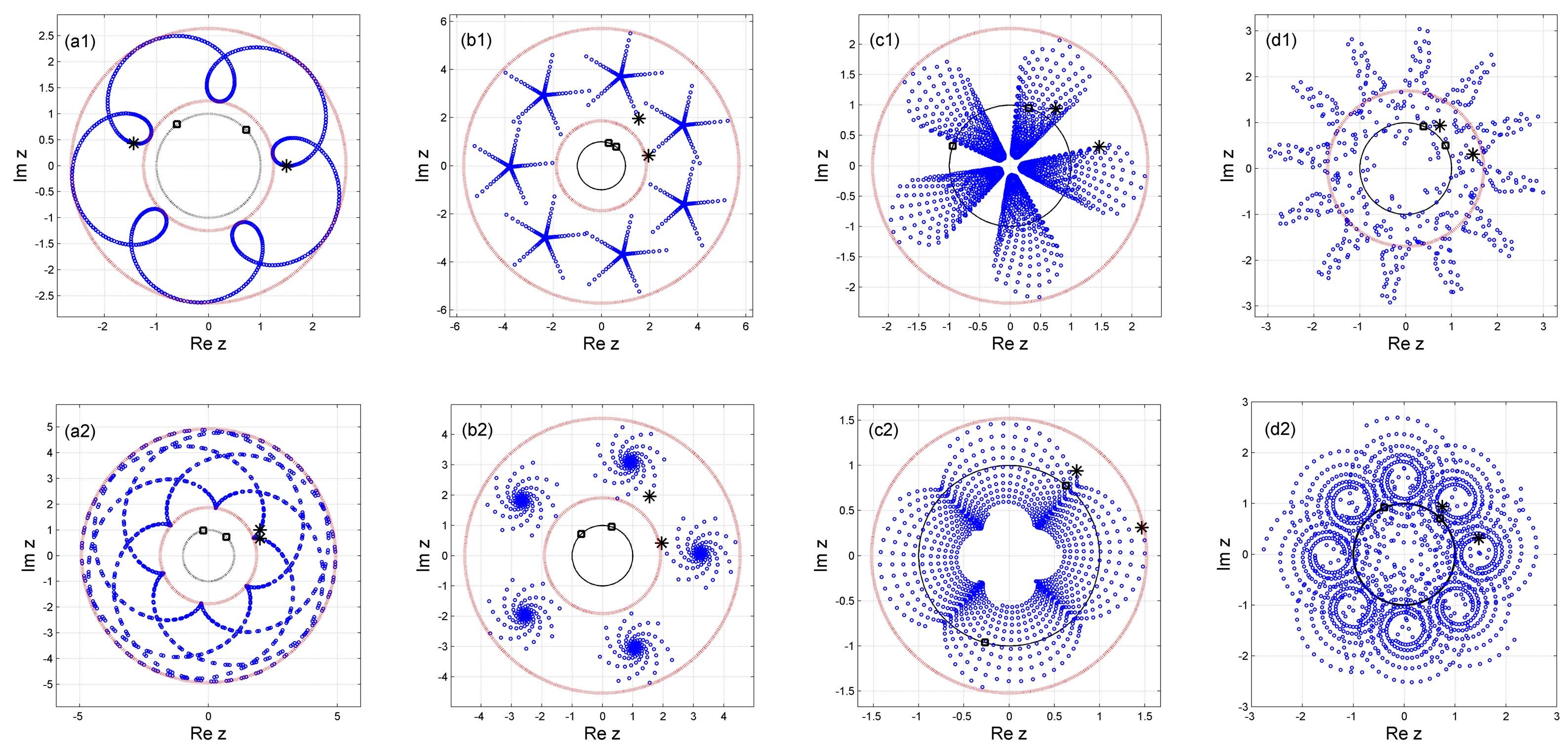


Figure 3: Horadam orbits: (a) Stable; (b) Quasi-convergent; (c) Convergent; (d) Divergent.

Dense Horadam orbits If $r_1 = r_2 = 1$ and the generators $z_1 = e^{2\pi i x_1} \neq z_2 = e^{2\pi i x_2}$ satisfy the relation $x_2 = x_1 q$ with $x_1, x_2, q \in \mathbb{R} \setminus \mathbb{Q}$, then the orbit of the Horadam sequence $\{w_n\}_{n=0}^{\infty}$ is dense in annulus $U(0, ||A| - |B||, |A| + |B|)$.

A Horadam-based pseudo-random number generator

Pseudo-random number generators Key features

- Requirements: period, uniformity, correlation
- Applications: numerical algorithms, simulations
- Implementation: Recurrences, Lagged Fibonacci, Mersenne Twister

Properties of Horadam sequence arguments. If $A = Re^{i\phi_1}$, $B = Re^{i\phi_2}$ one has

$$w_n = r_n e^{i\theta_n} = Az_1^n + Bz_2^n = Re^{i\left[\frac{\phi_1+\phi_2}{2} + 2\pi n(x_1+x_2)\right]} \tag{10}$$

The argument θ_n has the following properties:

- Aperiodicity: for x_1, x_2 irrational/uncorrelated, θ_n is aperiodic in $[-\pi, \pi]$
- Uniformity: for x_1, x_2 irrational/uncorrelated, $\tilde{\theta}_n = \frac{\theta_n + \pi}{2\pi}$ is uniform in $[0, 1]$
- Autocorrelation: normalized arguments $(\tilde{\theta}_n, \tilde{\theta}_{n+1})$ are correlated (linear)

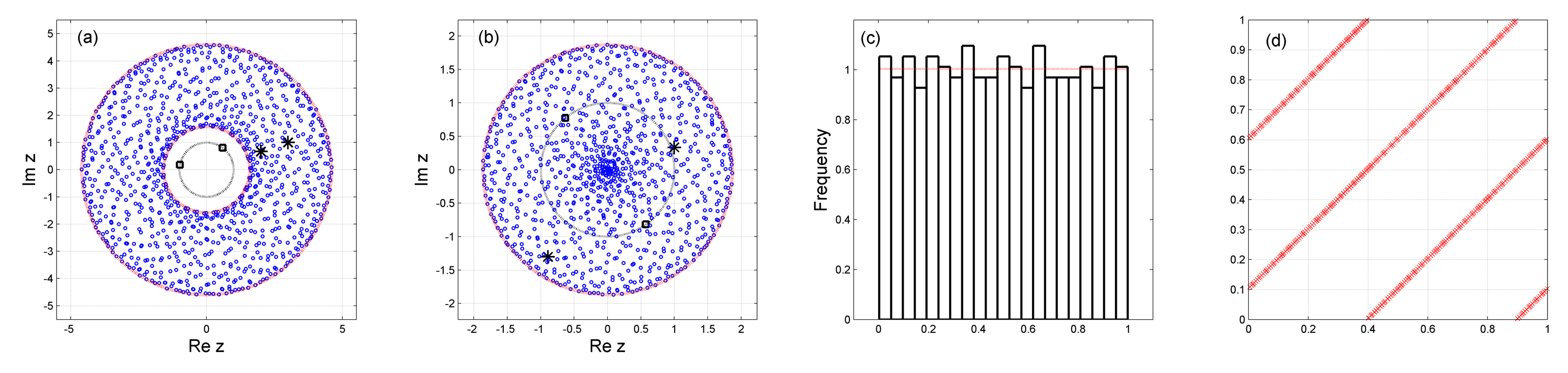
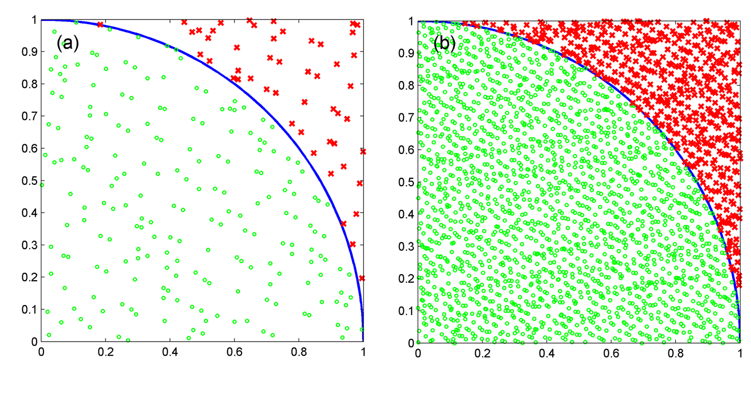


Figure 4: Dense Horadam sequence patterns obtained for (a) $|A| \neq |B|$ and (b) $|A| = |B|$. (c) Histogram of normalized angles $\tilde{\theta}_n$; (d) Correlation of arguments $(\tilde{\theta}_n, \tilde{\theta}_{n+1})$.

Monte Carlo simulations The value of π can be simulated as follows

- Take two dense Horadam sequences $\{w_n^1\}$ and $\{w_n^2\}$ $(r_1 = r_2 = 1)$
- Define $2D$ coordinates as $(x_n, y_n) = \left(\frac{\text{Arg}(w_n^1)+\pi}{2\pi}, \frac{\text{Arg}(w_n^2)+\pi}{2\pi}\right)$
- find m - the number of points satisfying $x_n^2 + y_n^2 \leq 1$
- determine the ratio $\rho = m/N$



(c) Table : 10^N is the sample size used in each simulation.

10^N	H1	H2	F1	F2	MT1	MT2
1	0.0584	0.0584	-0.3297	-0.3297	-0.7258	0.8584
2	0.2584	-0.0216	0.0985	-0.0615	-0.0215	0.0985
3	0.0784	-0.0016	-0.0136	0.0304	-0.0456	0.0264
4	0.0104	0.0004	0.0092	-0.0200	0.0036	0.0096
5	0.0012	-0.0006	-0.0016	-0.0018	0.0004	-0.0034
6	0.0003	0.0000	-0.0001	-0.0010	-0.0026	-0.0015
7	0.0000	0.0000	0.0003	-0.0006	-0.0002	0.0004

Figure 5: Monte Carlo simulation: (a) $N = 1000$, $\rho = 3.168$; (b) $N = 10000$, $\rho = 3.1420$; (c) Evaluation of results against Lagged Fibonacci and Mersenne Twister generators.

Conclusion and future work

The number and geometry of periodic Horadam sequences were presented. Non-periodic patterns were used to design a pseudo-random number generator. The results can be extended for generalized complex Horadam sequences.

References

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