Appl. Math. Lett. Vol. 2, No. 4, pp. 319–320, 1989
Printed in Great Britain.

0893-9659/89 \$3.00 + 0.00

A101858
→ A101330
A101866
A295573

Some Remarks About Fibonacci Multiplication

PIERRE ARNOUX
Dept. of Mathematics, University of Florida, Gainesville

Abstract. We give an interpretation of the Fibonacci multiplication defined by D.E. Knuth

1. Introduction.

It is well known and easy to prove that every nonnegative integer can be expressed in a unique way as a sum of Fibonacci numbers, with no two consecutive Fibonacci number in the sum; if we define $F_0 = 0$, $F_1 = 1$, $F_i + F_{i+1} = F_{i+2}$, the sum can be written $n = \sum_{i=0}^{N} d_i F_i$, with $d_i = 0$ or 1, $d_0 = d_1 = 0$, $d_i d_{i+1} = 0$ for all i.

In [K], Knuth defines "circle multiplication" by the following formula: if $n = \sum_{i=0}^{N} d_i F_i$ and $m = \sum_{j=0}^{M} e_j F_j$, then:

$$n \circ m = \sum_{i=0}^{N} \sum_{j=0}^{M} d_i e_j F_{i+j}$$

A101330

and he proves directly that this operation is associative. We will show that this multiplication can be interpreted as the usual multiplication in a multiplicatively closed subset of the ring $\mathbb{Z}[\phi]$ of algebraic integers generated by the "golden ratio" $\phi = (1+\sqrt{5})/2$, which implies the associativity; the "star product" $m \star n = mn + \lfloor \phi m \rfloor \lfloor \phi n \rfloor$, where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x, can be interpreted in the same way.

This proof of associativity is not really simpler that the one given in [K], because we have to use the same kind of combinatorics, but it makes the definition more natural, has interesting geometric properties, and is easy to generalize.

2. An algebraic version of Fibonacci multiplication.

We shall note Z the set of Zeckendorf representations, that is the set of finite sequences $(d_0, d_1, \dots, d_n)$ of 0 and 1 with $d_0 = d_1 = 0$ and $d_i d_{i+1} = 0$, as above; we note by $(d_0, d_1, \dots, d_n)_F = \sum_{i=0}^{n} d_i F_i$ the corresponding integer.

The Fibonacci multiplication makes sense on the elements of Z; if we associate to the finite sequence $(d_0, d_1, \dots, d_n)$ the polynomial $(d_0, d_1, \dots, d_n)_X = \sum_{i=0}^{n} d_i X^i$, it corresponds to the ordinary multiplication, modulo the rule $X^i + X^{i+1} = X^{i+2}$, which is the only property used to simplify the product. The Fibonacci multiplication thus appears as product in $\mathbb{Z}[X]/ < X^2 - X - 1 >$, and the following two lemmas suffice to prove associativity:

LEMMA 1. *Any element of* $\mathbb{Z}[X]/ < X^2 - X - 1 >$ *has at most one expression of the form* $(d_0, d_1, \dots, d_n)_X$, *with* $(d_0, d_1, \dots, d_n)$ *an element of* Z.

This proves that there is a well-defined correspondence between integers and polynomials of this form.

PROOF: We first remark the identity $X^n = F_{n-1} + F_n X$, which is easily proved by recurrence; it is then clear that, modulo $X^2 - X - 1$, we have $(d_0, d_1, \dots, d_n)_X = (d_1, \dots, d_n)_F + (d_0, d_1, \dots, d_n)_F X$; the lemma is then a consequence of the unicity of the Zeckendorf representation.($(d_1, \dots, d_n)_F$ is well defined, even if this sequence is not always in Z; the number so defined may have another representation, but this is unimportant, since we have unicity on the second coefficient.)

This paper was written while the author was visiting University of Florida, Gainesville

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

LEMMA 2. *The image of Z in $\mathbf{Z}[X]/ < X^2 - X - 1 >$ is closed under multiplication.*

We do not give a direct proof of this lemma; it can be made using the combinatorial methods of [K]; we give an alternate proof in the following section.

3. A real numbers version.

We call α the conjugate $(1-\sqrt{5})/2$ of the "golden ratio" $\phi = (1+\sqrt{5})/2$; we can define $(d_0, d_1, \ldots, d_n)_\alpha = \sum_{i=0}^{n} d_i \alpha^i$, and all the preceding section still works in this setting, since $\mathbf{Z}[\alpha]$ and $\mathbf{Z}[\phi]$ are the two realisations of $\mathbf{Z}[X]/ < X^2 - X - 1 >$ in $\mathbf{R}$; we have the following easy lemma:

LEMMA 3. *For any sequence $(d_0, d_1, \ldots, d_n)$ in Z, $(d_0, d_1, \ldots, d_n)_\alpha$ is contained in the interval $(-\alpha^2, -\alpha)$.*

Keep in mind that $\alpha = -1/\phi$ is negative!

PROOF: Since α^{2i} is positive, and α^{2i+1} is negative, it is clear that $\sum_{i=1}^{\infty} \alpha^{2i} = -\alpha$ is an upper bound for $(d_0, d_1, \ldots, d_n)_\alpha$, and similarly $\sum_{i=1}^{\infty} \alpha^{2i+1} = -\alpha^2$ is a lower bound

This gives an easy proof of lemma 2: the image of Z in $\mathbf{Z}[\alpha]$ is exactly the set of numbers $p + n\alpha$, with p, n nonnegative integers, which are contained in $(-\alpha^2, -\alpha)$; it is clear that this set is multiplicatively closed.

We thus get another way to compute $n \circ m$: first compute the numbers p_n and p_m such that $p_n + n\alpha$ and $p_m + m\alpha$ are contained in $(-\alpha^2, -\alpha)$ (we can get p_n from the Zeckendorf representation of n, by shifting it one step to the left); then use the identity $(p_n + n\alpha)(p_m + m\alpha) = p_{n \circ m} + (n \circ m)\alpha$, so that $n \circ m = nm + p_n m + n p_m$. This explains a remark one makes when computing a few examples: Fibonacci multiplication is almost distributive; in fact, the identity $(n+m) \circ l = n \circ l + m \circ l$ holds if and only if $(p_n + n\alpha) + (p_m + m\alpha)$ is still in $(-\alpha^2, -\alpha)$, which happens almost three times out of four on the average

This version of the Fibonacci multiplication has an interesting geometric interpretation: if we identify the circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$ with the interval $[-\alpha^2, -\alpha)$, we can define a multiplication on this circle; considering the rotation $R_\alpha : x \rightarrow x + \alpha \mod 1$ on the circle, we obtain that the positive orbit of 0 under this map is closed under this multiplication, with the rule: $R_\alpha^n(0).R_\alpha^m(0) = R_\alpha^{n \circ m}(0)$

A101858

4. The star-product of Porta and Stolarsky.

This operation is defined by: $n \star m = nm + \lfloor \phi n \rfloor \lfloor \phi m \rfloor$; it is similar to Fibonacci multiplication, and can also be defined using numbers in $\mathbf{Z}(\alpha)$, except that the emphasis is here on the integer component instead of the α-component, which explains why the star-product is asymptotically ϕ times greater than Fibonacci multiplication. Namely, for given n, find the unique integer k_n such that $0 \leq n + k_n\alpha < -\alpha$ (we have immediately $k_n = \lfloor \phi n \rfloor$); then we obtain easily the formula:

$$(n + \lfloor \phi n \rfloor \alpha)(m + \lfloor \phi n \rfloor \alpha) = n \star m + \lfloor \phi(n \star m) \rfloor \alpha$$

which immediately proves that this star-product is associative.

One can define an infinite number of such multiplications, by chosing another multiplicatively closed subset of $\mathbf{Z}[\alpha]$of length 1 or α; for example we could chose the elements of $\mathbf{Z}[\alpha]$ between 0 and 1, which would define another associative multiplication:

A101866

$$m \bullet n = mn + \left\lceil \frac{n}{\phi} \right\rceil \left\lceil \frac{m}{\phi} \right\rceil$$

where $\lceil x \rceil$ is the smallest integer not smaller than X.

This type of operation seems likely to generalize to other algebraic numbers; it would be particularly interesting for numbers of degree 3 or more.

REFERENCES

[K] D. E. Knuth, *Fibonacci multiplication*, Appl. Math. Lett. 1 (1988), 57–60.

Department of Mathematics, 201 Walker Hall, University of Florida, Gainesville FL32611, USA and Département de mathématiques, Université Paris-Sud, bâtiment 425, 91405 Orsay Cedex, FRANCE