Numbers N such that each appears at the N^{th} decimal place of 1/N

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Numbers that appear in the decimal expansion of their reciprocals 1/N are listed in A100468 in the ***On-Line Encyclopedia of Integer Sequences***. The requirement that the Number N also appears exactly at its Nth decimal place is a subset of A100468. A 2^{nd} subset is N at the N^{th} place not counting ***lead zeros***.

First to generate A100468 we have two conditions:

1. The required residual at the place where N appears in the decimal expansion is: rqres = $INT(N^2/10^{dg}) + 1$ and $[rqres*10^{dg} - N^2] < N$
2. For N even, we cannot have an odd rqres.

This generates the list: 3,6,7,14,17,28,58,59,83,86,87,89,97,118,...

NOTE: 1,10,100 are trivial cases: .1, .010, .00100, etc.

For say N = 17 we have reqres = $INT(17^2/100) + 1 = 3$, $3*10^2 - 17^2 = 11 < 17$ OK. Thus dividing 300 by 17 yields 17 + (residual < 17). To compute the residuals we use the MOD function repeatedly out to 2*N:

dg = 2, 10 ≡ 10 mod 17, 15 ≡ 10^2 mod 17, 14 ≡ 150 mod 17, 4 ≡ 140 mod 17, 6 ≡ 40 mod 17, 9 ≡ 60 mod 17, etc. ... when we get to the 11^{th} step we have 3 ≡ 20 mod 17 and 3 ≡ rqres. The decimal place = 11 + dg - 1 = 12^{th} place. Had there been no appearance of the rqres, or an even N with odd rqres, then that number isn't in A100468.

1/17 = period = 16

11th Step

.0588235294117647058823529411764705882352941176470588235294117

12th place

Now things get a bit more difficult. To have N appear <u>exactly</u> at the N^{th} place, we compute the "first appearance" **n** and the decimal period of 1/N = prd. In the above MOD function exercise, when we get to the rqres 3 at the 11th step, the first appearance n = 11+dg-1 = 12. This is repeated (even past N places to 2N) until we get another appearance of N at the rqres. (27^{th} step) The difference 27 - 11 = 16 is the period. You could also subtract 28 - 12 = 16.

In order to have N at the N^{th} place: n + k*period = N^{th} place or exactly or (N-n) ≡ 0 mod period or (N-n) ≡ (period -dg+1) mod period, without lead zeros.

Example: N = 228, n = 12, period = 18, 12 + k*18 = 228
k = (228 - 12) / 18 = 12 <u>exactly</u>. (k may not always = n) For say N = 97,
n = 39, period = 96 and k = (97 - 39) / 96 = .6041666 (not possible.)

n = 12

1/228 =
.00438596491**228**070175438596491**228**070175438596491**228**070175438596491
228070175438596491**228**070175438596491**228**070175438596491**228**07017543
8596491**228**070175438596491**228**070175438596491**228**070175438596491**228**0
70175438596491**228**070175438596491**228**

228^{th} place

We have 1, 3, 6, 228, ...

The second set (No lead zeros): 1, 3, 6, 14, (1,3,6 appear in <u>BOTH</u> Subsets)

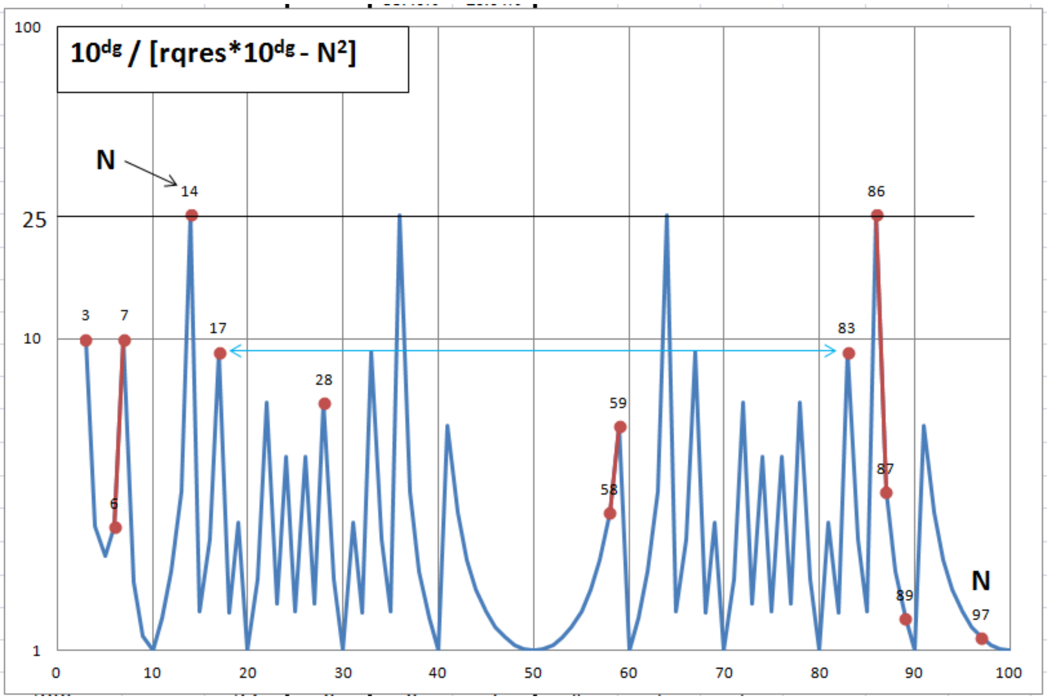
For 14, dg = 2, n = 3, period = 6, (14 - 3) = 5 mod 6

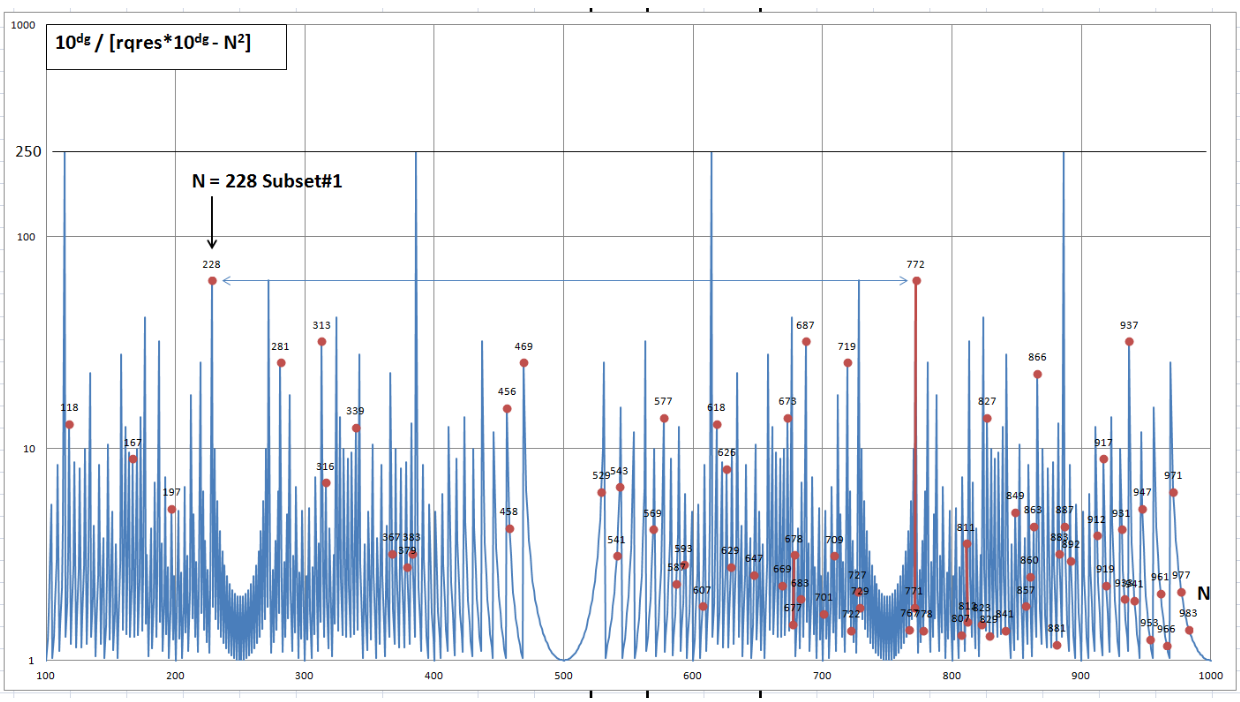
n = 3 15^{th} place period 6

1/14 = 0.07**14**2857**14**2857**14**2857**14**2857**14**2857...

14^{th} place = 15^{th} place less 1 zero

Plots of: N vs $10^{dg}/[rqres*10^{dg} - N^2]$ reveals some symmetry:

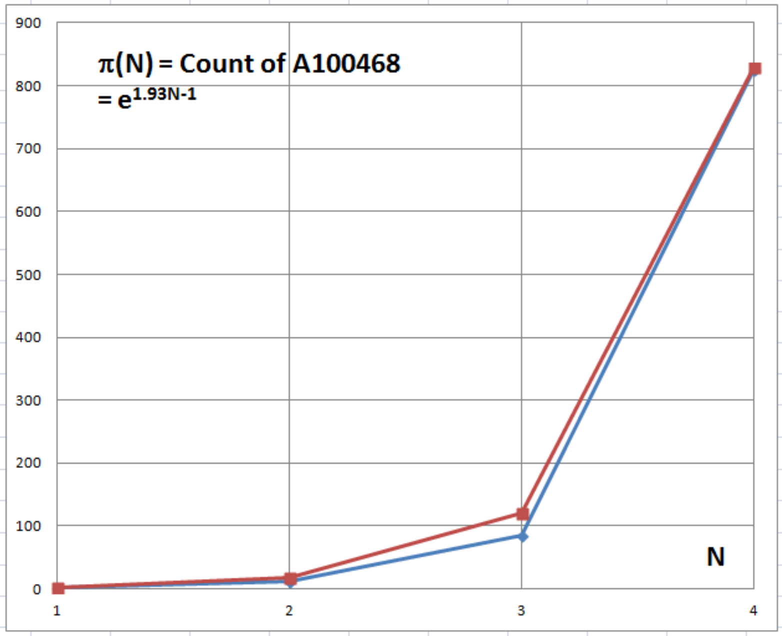




Define $\pi(N)$ similar to the prime counting function:

		m	b	
	0.995397	1.93	-1	
dg	Ln(π(x))		π(x)	
1	1.098612	0.93	3	2.534509
2	2.564949	2.86	13	17.46153
3	4.442651	4.79	85	120.3014
4	6.715383	6.72	825	828.8175

NOTE: OEIS has 825 + 4 = 829 up to 10,000
(1,10,100,1000) = 4 not included here



Basic Code

```
   ' find A100468 and subsets
   dim fa(10)
   for k = 1 to 300
   a$ = " "
   fa(1) = 0
   fa(2) = 0
   cc = 1
   n = k
   if n mod 1000 = 0 then print n
   i = 1
   dg = int(log(n)/log(10)+1)
   num = 10^(dg-1)
   reqres = int(n^2/10^dg)+1  'Required Residue
   test = reqres*10^(dg) - n^2   ' test #1
   if test >= n then goto 35
   if n mod 2 = 0 then gosub 100   ' test #2
25 resd = num mod n
   if resd = reqres then fa(cc) = i
   if resd = reqres then cc = cc +1
   if cc > 2 then perd = fa(2)-fa(1)
   if cc > 2 then goto 30
   num = 10*resd
   i = i + 1
   if i < 2*n then goto 25
30 frst = fa(1)+dg-1
   if frst < dg then goto 35
   if perd = 0 then goto 35
   test3 = (n-frst) mod perd
   if test3 > n then goto 150
   if test3 = 0 then a$ = "Subset#1"
   if test3 = perd - dg + 1 then a$ = "Subset#2"
33 Print "N=";n;tab(12);"n= ";frst;tab(20);"  Period=
";perd;tab(35);"N-n mod Period=";test3;
   Print tab(55);a$
35 next k
   print "End"
   end
100 if reqres mod 2 = 1 then goto 35
    return
150 if test3 = perd-dg+1 then goto 33
    goto 35
```

N=3	n= 1	Period= 1	N-n mod Period=0	Subset#1
N=6	n= 2	Period= 1	N-n mod Period=0	Subset#1
N=7	n= 6	Period= 6	N-n mod Period=1	
N=14	n= 3	Period= 6	N-n mod Period=5	Subset#2
N=17	n= 12	Period= 16	N-n mod Period=5	

```
N=28       n= 7     Period= 6    N-n mod Period=3
N=58       n= 19    Period= 28   N-n mod Period=11
N=59       n= 21    Period= 58   N-n mod Period=38
N=83       n= 35    Period= 41   N-n mod Period=7
N=86       n= 17    Period= 21   N-n mod Period=6
N=87       n= 10    Period= 28   N-n mod Period=21
N=89       n= 22    Period= 44   N-n mod Period=23
N=97       n= 39    Period= 96   N-n mod Period=58
N=118      n= 12    Period= 58   N-n mod Period=48
N=167      n= 103   Period= 166  N-n mod Period=64
N=197      n= 12    Period= 98   N-n mod Period=87
N=228      n= 12    Period= 18   N-n mod Period=0    Subset#1
N=281      n= 21    Period= 28   N-n mod Period=8
End
```

' find reciprocals at nth place (ONLY Subsets)

```
   dim fa(10)
   for k = 3 to 1000
   fa(1) = 0
   fa(2) = 0
   cc = 1
   n = k
   if n mod 100 = 0 then print n
   i = 1
   dg = int(log(n)/log(10)+1)
   num = 10^(dg-1)
   reqres = int(n^2/10^dg)+1  'Required Residue
   test = reqres*10^(dg) - n^2   ' test #1
   if test >= n then goto 35
   if n mod 2 = 0 then gosub 100   ' test #2
25 resd = num mod n
   if resd = reqres then fa(cc) = i
   if resd = reqres then cc = cc +1
   if cc > 2 then perd = fa(2)-fa(1)
   if cc > 2 then goto 30
   num = 10*resd
   i = i + 1
   if i < 2*n then goto 25
30 frst = fa(1)+dg-1
   if frst < dg then goto 35
   if perd = 0 then goto 35
   test3 = (n-frst) mod perd
   if test3 = 0 then goto 33
   if test3 <> perd-dg+1 then goto 35
33 Print n;" First= ";frst;"  Period= ";perd;" Mod test= ";test3
35 next k
   print "End"
   end
100 if reqres mod 2 = 1 then goto 35    ' Even N test
    return
150 if test3 > perd - dg +1 then goto 33    ' Lead Zero test
```

```
    goto 35

3 First= 1  Period= 1 Mod test= 0
6 First= 2  Period= 1 Mod test= 0
14 First= 3  Period= 6 Mod test= 5
100
200
228 First= 12  Period= 18 Mod test= 0
300
400
500
600
700
800
900
1000
End
```

```
   ' Print ALL A100468
   dim fa(10)
   pix = 0      ' A100468 COUNTER
   for k = 1 to 1001
   dg1 = (log(k)/log(10))
   if dg1 - int(dg1) < 1/10^dg then print " pix= ";pix  ' magnitude of
10
   if dg1 - int(dg1) < 1/10^dg then ct = 0   ' magnitude of 10
   a$ = " "
   fa(1) = 0
   fa(2) = 0
   n = k
   cc = 1
   i = 1
   dg = int(log(n)/log(10)+1)
   num = 10^(dg-1)
   reqres = int(n^2/10^dg)+1  'Required Residue
   test = reqres*10^(dg) - n^2   ' test #1
   if test >= n then goto 35
   if n mod 2 = 0 then gosub 100   ' test #2
25 resd = num mod n
   if resd = reqres then fa(cc) = i
   if resd = reqres then cc = cc +1
   if cc > 2 then perd = fa(2)-fa(1)
   if cc > 2 then goto 30
   num = 10*resd
   i = i + 1
   if i < 2*n then goto 25
30 frst = fa(1)+dg-1
   if frst < dg then goto 35
   if perd = 0 then goto 35
   test3 = (n-frst) mod perd
   if test3 > n then goto 150
```

```
   if test3 = 0 then a$ = "Subset#1"
   if test3 = perd - dg + 1 then a$ = "Subset#2"
33 Print n;" ";
   pix = pix + 1
   ct = ct + 1
   if ct mod 10 = 0 then print
35 next k
   print "End"
   end
100 if reqres mod 2 = 1 then goto 35
    return
150 if test3 = perd-dg+1 then goto 33
    goto 35
```

First 825 up to 10,000 (NOTE 1,10,100,1000,10000 not counted add 5)
OEIS has #825 + 5 = #830 = 10,000 (same)
#825 + 4 = #829 = 9979 OEIS

3 6 7 pix= 3 up to 10 (#4 OEIS = 7 add 1)

14 17 28 58 59 83 86 87 89 97 pix= 13 (#15 = 97 OEIS add 1,10)

118 167 197 228 281 313 316 339 367 379
383 456 458 469 529 541 543 569 577 587
593 607 618 626 629 647 669 673 677 678
683 687 701 709 719 722 727 729 767 771
772 778 807 811 812 823 827 829 841 849
857 860 863 866 881 883 887 892 912 917
919 931 933 937 941 947 953 961 966 971
977 983 pix= 85 (#88 = 983 OEIS add 1,10,100)

1058 1063 1086 1109 1153 1349 1356 1367 1389 1459
1473 1483 1503 1523 1549 1565 1581 1661 1697 1726
1732 1766 1783 1811 1819 1838 1841 1849 1873 1913
1949 1954 1977 1997 2017 2027 2083 2102 2126 2147
2177 2179 2209 2218 2227 2249 2269 2280 2293 2302
2317 2362 2383 2433 2437 2447 2557 2567 2617 2621
2634 2657 2687 2698 2707 2711 2731 2751 2753 2764
2789 2807 2839 2851 2867 2891 2893 2903 2917 2927
2939 2946 3013 3023 3043 3046 3098 3127 3130 3162
3167 3178 3203 3209 3251 3257 3274 3292 3391 3394
3401 3407 3413 3426 3435 3461 3464 3481 3521 3527
3544 3551 3583 3586 3593 3622 3637 3651 3659 3682
3701 3709 3752 3793 3826 3847 3857 3887 3901 3914
3924 3943 3947 3953 3967 3981 4007 4011 4057 4073
4079 4084 4091 4135 4142 4153 4177 4219 4226 4233
4252 4254 4259 4261 4267 4294 4337 4339 4354 4363
4393 4418 4427 4436 4447 4454 4463 4471 4481 4491
4523 4567 4591 4593 4604 4629 4673 4685 4687 4690
4691 4703 4722 4723 4724 4759 4783 4802 4808 4811
4863 4866 4873 4874 4877 4909 4919 4927 5087 5099
5126 5127 5129 5134 5141 5167 5188 5189 5191 5192

5198	5223	5242	5274	5277	5281	5297	5309	5311	5314
5315	5331	5367	5374	5381	5396	5398	5407	5419	5422
5444	5446	5447	5489	5506	5507	5519	5524	5527	5528
5545	5563	5573	5581	5589	5613	5614	5623	5647	5662
5669	5678	5689	5734	5741	5753	5774	5779	5787	5834
5851	5857	5869	5881	5892	5894	5897	5899	5903	5914
5927	5938	5943	5947	5953	5978	5987	5989	6029	6038
6046	6047	6053	6063	6067	6086	6103	6109	6113	6117
6121	6131	6143	6166	6173	6181	6199	6207	6211	6227
6231	6243	6247	6263	6267	6287	6302	6313	6317	6326
6334	6337	6356	6367	6389	6406	6412	6414	6423	6428
6431	6434	6442	6449	6454	6456	6465	6473	6502	6508
6509	6515	6536	6551	6571	6581	6583	6587	6593	6598
6599	6603	6621	6657	6661	6673	6691	6693	6701	6737
6742	6745	6766	6769	6779	6782	6788	6793	6821	6822
6823	6826	6827	6829	6833	6838	6857	6859	6861	6869
6870	6883	6893	6899	6918	6922	6928	6938	6949	6967
6971	6977	6991	6994	6997	7017	7019	7027	7031	7048
7054	7078	7088	7109	7126	7127	7153	7162	7167	7177
7181	7183	7186	7187	7190	7193	7219	7221	7222	7229
7235	7237	7244	7247	7262	7269	7281	7291	7297	7301
7302	7318	7327	7331	7349	7351	7361	7365	7393	7411
7433	7451	7457	7459	7477	7481	7487	7489	7493	7499
7517	7523	7529	7531	7541	7547	7559	7591	7603	7607
7622	7635	7647	7652	7653	7655	7673	7679	7687	7694
7699	7703	7720	7721	7727	7751	7753	7764	7765	7778
7779	7783	7791	7806	7814	7817	7823	7829	7837	7867
7873	7879	7894	7897	7898	7901	7915	7921	7927	7932
7937	7939	7949	7962	7966	7972	7973	7981	7983	8017
8022	8039	8057	8059	8062	8069	8087	8089	8114	8117
8123	8139	8146	8147	8158	8168	8171	8179	8185	8207
8209	8219	8231	8233	8263	8267	8269	8270	8273	8287
8298	8309	8311	8339	8347	8352	8353	8371	8377	8389
8419	8423	8429	8431	8434	8438	8443	8447	8452	8458
8461	8466	8485	8486	8501	8504	8506	8518	8519	8521
8535	8543	8572	8573	8581	8588	8594	8623	8641	8654
8660	8663	8669	8674	8683	8698	8699	8707	8709	8713
8717	8733	8741	8753	8781	8786	8793	8807	8809	8813
8817	8819	8831	8836	8846	8854	8859	8861	8863	8871
8872	8878	8885	8887	8903	8908	8913	8914	8926	8942
8947	8954	8962	8971	8984	8997	9004	9011	9019	9029
9043	9047	9053	9055	9057	9059	9062	9101	9103	9107
9109	9122	9137	9143	9149	9166	9187	9203	9221	9233
9239	9251	9252	9257	9283	9294	9311	9319	9341	9343
9349	9368	9370	9371	9374	9377	9387	9389	9391	9406
9409	9411	9413	9415	9421	9454	9457	9467	9473	9478
9479	9487	9491	9497	9498	9518	9523	9539	9547	9556
9557	9558	9569	9571	9586	9587	9604	9611	9617	9619
9623	9629	9634	9659	9662	9673	9679	9686	9697	9709
9719	9726	9732	9739	9743	9749	9758	9761	9767	9769
9777	9781	9787	9793	9802	9803	9811	9817	9818	9829
9833	9851	9853	9857	9861	9862	9866	9873	9874	9882

9886 9887 9893 9897 9913 9923 9931 9949 9967 9979

pix= 825 End (#829 = 9979 OEIS add 1,10,100,1000)