Notes on the constants A096427 and A224268

Peter Bala, Feb 2019

We define three constants

$$C_1 = \prod_{n=1}^{\infty}\left(1-\frac{1}{(4n+1)^2}\right) = 0.92703\,73386...$$

$$C_2 = \prod_{n=0}^{\infty}\left(1-\frac{1}{(4n+2)^2}\right) = 0.70710\,67811...$$

$$C_3 = \prod_{n=0}^{\infty}\left(1-\frac{1}{(4n+3)^2}\right) = 0.84721\,30847...$$

These constants, and several related constants, are recorded in the OEIS. C_1 is A224268. $2C_1$ is A093341. It follows easily from Euler's infinite product representation for the cosine function that C_2 is equal to $\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, entry A010503. The constant C_3 is A096427; its reciprocal $1/C_3$ is A175574. The constant C_3^2 is A293238; its reciprocal $1/C_3^2$ is A091670, the value of the first of Watson's triple integrals. The constant $8C_1C_2$ is A064853, the perimeter of the lemniscate curve $r = \cos(2\theta)$. The constant $2C_1/C_3$ is A254794.

Our interest in the constants C_1 and C_3 arose when they appeared in some hypergeometric sums we were investigating. This is explained in Section 2. First we list some results for the constants C_1 and C_3.

Relationship between C_1 and C_3

$$\boxed{C_1C_3 = \frac{\pi}{4}} \tag{1}$$

Proof. The rearrangement of terms in the following is justified since the infinite products involved converge absolutely.

$$\begin{aligned} C_1C_3 &= \prod_{n=1}^{\infty}\left(1-\frac{1}{(4n+1)^2}\right)\prod_{n=0}^{\infty}\left(1-\frac{1}{(4n+3)^2}\right) \\ &= \prod_{n=1}^{\infty}\left(1-\frac{1}{(4n-1)^2}\right)\left(1-\frac{1}{(4n+1)^2}\right) \\ &= \prod_{n=1}^{\infty}\left(1-\frac{1}{(2n+1)^2}\right). \end{aligned} \tag{2}$$

We evaluate the latter product by means of Euler's infinite product representation for the cosine function:

$$\cos(\pi x) \quad = \quad \prod_{n=0}^{\infty}\left(1-\frac{4x^2}{(2n+1)^2}\right), x \in \mathbb{C},$$

rearranged into the form

$$\frac{\cos(\pi x)}{1-4x^2} \quad = \quad \prod_{n=1}^{\infty}\left(1-\frac{4x^2}{(2n+1)^2}\right).$$

Let now x approach $\frac{1}{2}$ and use L'Hôpital's rule to find

$$\begin{aligned}\frac{\pi}{4} &= \prod_{n=1}^{\infty}\left(1-\frac{1}{(2n+1)^2}\right) \qquad (3)\\ &= C_1 C_3\end{aligned}$$

by (2). ■

Representation in terms of the gamma function

$$C_1 \quad = \quad \frac{\Gamma\left(\frac{1}{4}\right)^2}{8\sqrt{\pi}} \qquad (4)$$

$$C_3 \quad = \quad \frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{\pi}} \qquad (5)$$

These well-known results are proved in Appendix A, along with the proof for the following continued fraction expansions.

Generalised continued fraction expansions

C_1 :

$$1-\frac{1}{5+}\;\frac{20}{1+}\;\frac{30}{3+}\;\cdots+\;\frac{4n(4n+1)}{1\quad+}\;\frac{(4n+1)(4n+2)}{3\quad+}\;\cdots \qquad (6)$$

C_3 :

$$1-\frac{1}{3+}\;\frac{6}{1+}\;\frac{12}{3+}\;\cdots+\;\frac{(4n-2)(4n-1)}{1\quad+}\;\frac{(4n-1)(4n)}{3\quad+}\;\cdots \qquad (7)$$

$\frac{C_1}{C_3}$:

$$\frac{1}{2}\left(1+\frac{2}{1+}\;\frac{3}{2+}\;\frac{15}{2+}\;\cdots+\;\frac{(2n-1)(2n+1)}{2\quad+}\;\cdots\right) \tag{8}$$

Infinite products

For $m = 0, 1, 2, ...$,

$$C_1 = \frac{(-1)^m 2^{2m+1}}{\text{Catalan}(m)} \prod_{n=1}^{\infty}\left(1-\frac{(4m+3)^2}{(4n+1)^2}\right) \tag{9}$$

$$C_3 = \frac{(-1)^m 2^{2m}}{\binom{2m}{m}} \prod_{n=0}^{\infty}\left(1-\frac{(4m+1)^2}{(4n+3)^2}\right) \tag{10}$$

For $m = 0, 1, 2, ...$,

$$\frac{C_1}{C_3} = -\frac{1}{2}\prod_{k=1}^{m}\frac{(1-4k)}{(1+4k)} \prod_{n=0}^{\infty}\left(1-\frac{(4m+2)^2}{(4n+1)^2}\right) \tag{11}$$

$$\frac{C_3}{C_1} = 2\prod_{k=1}^{m}\frac{(1+4k)}{(1-4k)} \prod_{n=0}^{\infty}\left(1-\frac{(4m+2)^2}{(4n+3)^2}\right) \tag{12}$$

Series representations

$$C_1 = \sum_{n=0}^{\infty}\frac{\binom{-\frac{1}{2}}{n}}{4n+1}; \quad C_1 = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n\binom{-\frac{3}{4}}{n}}{4n+1} \tag{13}$$

$$C_3 = 1-\sum_{n=0}^{\infty}\frac{\binom{\frac{1}{2}}{n+1}}{4n+3} \tag{14}$$

These results are particular cases of two product-to-sum identities proved in Section 2.2, Theorem 1.

Integral representations

$$C_1 = \int_0^1 \frac{dx}{\sqrt{1+x^4}} \tag{15}$$

$$C_3 = \int_0^1 \frac{1+x^2-\sqrt{1+x^4}}{x^2}\,dx \tag{16}$$

These integral representions are equivalent to the corresponding series representations immediately above. To see this, we expand the integrands using the binomial theorem and then integrate the resulting series term by term from 0 to $u < 1$ (allowable, since we are in the region of uniform convergence of the series). We then invoke Abel's lemma to justify setting $u = 1$ in the results, when the first series in (13) and the series in (14) are obtained.

Theta function representations

$$C_1 = \frac{\pi}{4}\left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2}\right)^2 \tag{17}$$

$$C_3 = \frac{1}{\left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2}\right)^2} \tag{18}$$

Conjecturally,

$$C_3 = \frac{1}{\left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi\left(n+\frac{1}{2}\right)^2}\right)^2} \tag{19}$$

For (18), see Section 7 in the Wikipedia entry - Theta function.

Section 2

In Section 2.1 we sketch a method for representing the infinite product constant $\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n+1)^2}\right)$ as a series. In Section 2.2 we follow a similar path to find the series representations for the infinite products C_1 and C_3 stated in (13) and (14).

2.1 Ramanujan gave the following hypergeometric summation [Berndt, Example 8, p. 21]:

$$1-3\frac{(x-1)}{(x+1)}+5\frac{(x-1)(x-2)}{(x+1)(x+2)}-\cdots \quad = \quad 0, \quad \mathrm{Re}(x)>1. \tag{20}$$

In [Bala], we investigated hypergeometric series $S_r(x), r \in \mathbb{Z}$ defined as

$$S_r(x) \quad = \quad 1-3^r\frac{(x-1)}{(x+1)}+5^r\frac{(x-1)(x-2)}{(x+1)(x+2)}-\cdots.$$

Ramanujan's result (20) says that the series $S_1(x)$ vanishes identically for $\mathrm{Re}(x)>1$.

Let $a_k(x)$ denote the rational function of x

$$a_k(x)=\frac{(x-1)(x-2)\ldots(x-k)}{(x+1)(x+2)\ldots(x+k)}, \quad k=1,2,3,...,$$

with $a_0(x)=1$, so that

$$S_r(x) \quad = \quad \sum_{k=0}^{\infty}(-1)^k(2k+1)^r a_k(x).$$

It is easy to verify that the function $a_k(x)$ satisfies the recurrence equation (in x)

$$(2k+1)^2 a_k(x) \quad = \quad (2x-1)^2 a_k(x)-4x(x-1)a_k(x-1). \tag{21}$$

We immediately obtain the corresponding recurrence for the series $S_r(x)$ (putting aside issues of convergence):

$$S_{r+2}(x) \quad = \quad (2x-1)^2 S_r(x)-4x(x-1)S_r(x-1). \tag{22}$$

In particular, this recurrence holds when $x=n$, a positive integer, because in this case all the series $S_r(n)$ terminate since $a_k(n)=0$ for $k \geq n$.

Setting $r = -1$ in the recurrence (22) leads, after a short calculation, to a product representation for the series $S_{-1}(n)$:

$$1 - \frac{1}{3}\frac{(n-1)}{(n+1)} + \frac{1}{5}\frac{(n-1)(n-2)}{(n+1)(n+2)} - \cdots \quad = \quad \prod_{j=1}^{n-1}\left(1 - \frac{1}{(2j+1)^2}\right). \tag{23}$$

Now let n tend to ∞ on both sides of (23): with a little care [1] we can justify taking the limit term by term in the series on the left-hand side of (23) to obtain the series - to - product identity

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots \quad = \quad \prod_{j=1}^{\infty}\left(1 - \frac{1}{(2j+1)^2}\right). \tag{24}$$

By (3), this result is equivalent to the famous Madhava - Leibniz series for $\frac{\pi}{4}$.

2.2 Series representations for C_1 and C_3

Let now $b_n(x)$ denote the rational function in x

$$b_k(x) = \frac{(x-1)(x-2)\cdots(x-k)}{\left(x+1-\frac{1}{2}\right)\left(x+2-\frac{1}{2}\right)\cdots\left(x+k-\frac{1}{2}\right)} \quad k = 1, 2, 3, \ldots . \tag{25}$$

It is not difficult to verify that the functions $b_k(x), k = 0, 1, 2, \ldots$, satisfy the following recurrence equation in x:

$$(4k+1)^2 b_k(x) \quad = \quad (4x-3)^2 b_k(x) - (4x-2)(4x-4)b_k(x-1). \tag{26}$$

Let $f(n)$ be (for the moment) an arbitrary arithmetical function and consider the series $S_r(f;x), r \in \mathbb{Z}$, defined by

$$S_r(f;x) \quad = \quad 1 + \sum_{k=1}^{\infty}(4k+1)^r f(k) b_k(x). \tag{27}$$

Setting aside questions of convergence, it follows immediately from (26) that the series $S_r(f;x)$ satisfies the recurrence

$$S_{r+2}(f;x) \quad = \quad (4x-3)^2 S_r(f;x) - (4x-2)(4x-4)S_r(f;x-1). \tag{28}$$

In particular, this recurrence will hold if x is a positive integer n, since in this case the series $S_r(f,n)$ terminate and questions of convergence do not arise.

[1] An example of the type of reasoning needed here can be found in Knopp's Theory and Application of Infinite Series, Dover Publ. 1990, §23, p. 193.

Following the development in Section 2.1, we look for an arithmetical function $f(n)$ such that the series

$$S_1(f;n) \quad = \quad 1 + 5f(1)b_1(n) + 9f(2)b_2(n) + \cdots \tag{29}$$

vanishes for all integer $n \geq 2$. The required values of $f(1), f(2), ...$ may be easily determined by successively setting $n = 2, 3, 4, ...$ in the terminating series (29) and equating the results to 0. After a short calculation we obtain the following results: $f(1) = -\frac{1}{2}$, $f(2) = \frac{3}{8}$, $f(3) = -\frac{5}{16}$, $f(4) = \frac{35}{128}$, $f(5) = -\frac{63}{256}$, $f(6) = \frac{231}{1024}$, $f(7) = -\frac{429}{2048}, ...$. A search of the OEIS for the sequence of (unsigned) numerators of these values returns a single potential match in A001790 - the sequence of numerators in the expansion of $1/\sqrt{1-x}$. This suggests that the correct choice for $f(n)$ to ensure that the series $S_1(f;n)$ vanishes is $f(n) = \binom{-\frac{1}{2}}{n}$. We now verify this. For the remainder of this section we take $f(n) = \binom{-\frac{1}{2}}{n}$.

Proposition 1. We have $S_1(f;n) = 0$ for $n = 2, 3, 4, ...$.

Proof. It is not difficult to inductively show that the p^{th} partial sum of the series $S_1(f;x)$ is equal to

$$2^p \binom{-\frac{3}{2}}{p} \frac{(x-2)(x-3)\cdots(x-(p+1))}{(2x+1)(2x+3)\cdots(2x+2p-1)}.$$

Therefore, for a fixed integer $n \geq 2$, the p^{th} partial sum of the series $S_1(f;n)$ is equal to zero provided $p \geq n-1$. Hence the series $S_1(f;n)$, as the limit of these partial sums, also has the value zero. ■

Corollary 1. *For $r = 0, 1, 2, ...$, and for integer $n \geq r+2$, the series $S_{2r+1}(f;n) = 0$.*

Follows immediately from Proposition 1 and the recurrence (28). ■

Proposition 2 *Let n be a positive integer. Then*

$$S_{-1}(f;n) \quad = \quad \prod_{k=1}^{n-1} \left(1 - \frac{1}{(4k+1)^2}\right). \tag{30}$$

Proof. The result clearly holds when $n = 1$ if we adopt the usual convention that empty products have the value 1. Suppose now that $n \geq 2$. Set $r = -1$ in the recurrence (28) and use Proposition 1 to find

$$0 \quad = \quad (4n-3)^2 S_{-1}(f;n) - (4n-2)(4n-4)S_{-1}(f;n-1), \quad n = 2, 3, 4, ...,$$

from which we get

$$\begin{aligned} S_{-1}(f;n) \quad &= \quad \frac{(4n-2)(4n-4)}{(4n-3)^2} S_{-1}(f;n-1) \\ &= \quad \left(1 - \frac{1}{(4n-3)^2}\right) S_{-1}(f;n-1). \end{aligned}$$

Iterating this result yields

$$S_{-1}(f;n) \quad = \quad \left(1-\frac{1}{(4n-3)^2}\right)\cdots\left(1-\frac{1}{5^2}\right)S_{-1}(f;1)$$

Clearly, $S_{-1}(f;1)=1$ from the definition of the series $S_r(f;x)$ in (27), and so we have completed the proof that

$$S_{-1}(f;n) \quad = \quad \prod_{k=1}^{n-1}\left(1-\frac{1}{(4k+1)^2}\right) \tag{31}$$

when n is a positive integer. ■

Corollary 2.

$$C_1=\prod_{k=1}^{\infty}\left(1-\frac{1}{(4k+1)^2}\right) \quad = \quad \sum_{n=0}^{\infty}\frac{\binom{-\frac{1}{2}}{n}}{4n+1}\,. \tag{32}$$

Proof. Let $n\to\infty$ on both sides of (31). Again, with a little care, we can justify letting $n\to\infty$ term by term in the series on the left side of (31). ■

We may view Corollary 2 as an analogue of the Madhava - Leibniz series for π :

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \quad = \quad \frac{\pi}{4}\,. \tag{33}$$

The analogy between (32) and (33) becomes clearer if we note that $(-1)^n=\binom{-1}{n}$ and then use (3) to recast the Madhava - Leibniz series into the form

$$\sum_{n=0}^{\infty}\frac{\binom{-1}{n}}{2n+1} \quad = \quad \prod_{k=1}^{\infty}\left(1-\frac{1}{(2k+1)^2}\right). \tag{34}$$

In fact the pair of results (32) and (34) are the particular cases $x=-\frac{1}{4}$ and $x=-\frac{1}{2}$ of part (i) of the following sum-to-product identities:

Theorem 1. (i) Let $\mathrm{Re}(x)>-1$ and $x\neq 0,1,2,\ldots$. Then

$$\sum_{n=0}^{\infty}\binom{2x}{n}\frac{x}{x-n} \quad = \quad \prod_{k=1}^{\infty}\left(1-\frac{x^2}{(k-x)^2}\right). \tag{35}$$

(ii) Let $\mathrm{Re}(x)<0$ and $x\neq 0,1,2,\ldots$. Then

$$\sum_{n=0}^{\infty}\binom{x+n}{n}\frac{x}{x-n} \quad = \quad 2\prod_{k=1}^{\infty}\left(1-\frac{x^2}{(k-x)^2}\right). \tag{36}$$

Proof. The series on the left-hand side of (35) is a Gaussian hypergeometric function:

$$\sum_{n=0}^{\infty}\binom{2x}{n}\frac{x}{x-n} \quad = \quad {}_2F_1(-2x,-x;1-x;-1).$$

By standard results in hypergeometric function theory the series converges provided $\mathrm{Re}(x)>-1$; the convergence is absolute when $\mathrm{Re}(x)>-\frac{1}{2}$.

Applying Kummer's non-terminating summation theorem [see, for example, W. N. Bailey, "Kummer's Theorem" §2.3 in Generalised Hypergeometric Series: Cambridge University Press, pp. 9-10, 1935]

$${}_2F_1(a,b;1+a-b;-1) \quad = \quad \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)}, \quad \mathrm{Re}(b)<1,$$

with $a=-2x$, $b=-x$ and $c=1-x$, where $\mathrm{Re}(x)>-1$, yields

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{2x}{n}\frac{x}{x-n} &= \frac{\Gamma(1-x)\Gamma\left(1-x\right)}{\Gamma(1-2x)\Gamma(1)} \\
&= \prod_{k=0}^{\infty}\left(\frac{(k+1)(k+1-2x)}{(k+1-x)(k+1-x)}\right) \\
&= \prod_{k=1}^{\infty}\left(\frac{k(k-2x)}{(k-x)^2}\right) \\
&= \prod_{k=1}^{\infty}\left(1-\frac{x^2}{(k-x)^2}\right).
\end{aligned}$$

The representation of the ratio of gamma functions as an infinite product used in the above follows easily from the Weierstrass product formula for the gamma function. See Appendix A, equation (45) .

(ii) The series on the left-hand side of (36) is a Gaussian hypergeometric function:

$$\sum_{n=0}^{\infty}\binom{x+n}{n}\frac{x}{x-n} \quad = \quad {}_2F_1(1+x,-x;1-x;1).$$

Again, by standard results in hypergeometric function theory, the hypergeometric series converges absolutely when $\mathrm{Re}(x)<0$.

Applying Gauss's theorem [see, for example, W. N. Bailey, "Gauss's Theorem" §1.3 in Generalised Hypergeometric Series: Cambridge University Press, pp. 2-3, 1935]

$${}_2F_1(a,b;c;1) \quad = \quad \frac{\Gamma(c)\Gamma\left(c-a-b\right)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(a+b-c)<0,$$

with $a = 1 + x$, $b = -x$ and $c = 1 - x$, where $\mathrm{Re}(x) < 0$, yields

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{x+n}{n} \frac{x}{x-n} &= \frac{\Gamma(1-x)\Gamma(-x)}{\Gamma(-2x)\Gamma(1)} \\ &= \frac{-2x}{-x} \frac{\Gamma(1-x)\Gamma(1-x)}{\Gamma(1-2x)\Gamma(1)} \\ &= 2 \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(k-x)^2}\right),\end{aligned}$$

by the calculation in part (i). ■

To obtain series representations for the constant C_1 put $x = -\frac{1}{4}$ in (35) and in (36) to give the pair of results stated in (13)

$$C_1 = \sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n}}{4n+1}, \quad C_1 = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{-\frac{3}{4}}{n}}{4n+1}.$$

To obtain a series representation for the constant C_3 put $x = \frac{1}{4}$ in (35) to find

$$\sum_{n=0}^{\infty} \frac{\binom{\frac{1}{2}}{n}}{1-4n} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{(4k-1)^2}\right),$$

or equivalently,

$$\begin{aligned}1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} \frac{1}{4n+3} &= \prod_{k=0}^{\infty} \left(1 - \frac{1}{(4k+3)^2}\right) \\ &= C_3.\end{aligned} \tag{37}$$

This is the series representation for the constant C_3 stated in (14). We give an alternative approach to Theorem 1 part (i) in Appendix B.

2.3 The series $S_{-3}(f,n)$, $S_{-5}(f,n), \ldots$.

Recall the series $S_r(f;x), r \in \mathbb{Z}$, was defined by

$$S_r(f;x) = 1 + \sum_{k=1}^{\infty} (4k+1)^r f(k) b_k(x),$$

where $f(n) = \binom{-\frac{1}{2}}{n}$. To abbreviate notation let us write

$$P(n) = \prod_{k=1}^{n-1} \left(1 - \frac{1}{(4k+1)^2}\right).$$

Proposition 2 tells us that $S_{-1}(f;n) = P(n)$ for positive integer n. Then putting $r = -3$ in the recurrence (28) gives

$$P(n) \quad = \quad (4n-3)^2 S_{-3}(f;n) - (4n-2)(4n-4)S_{-3}(f;n-1), \tag{38}$$

from which we get

$$S_{-3}(f;n) \quad = \quad P(n)\frac{1}{(4(n-1)+1)^2} + \left(1 - \frac{1}{(4(n-1)+1)^2}\right) S_{-3}(f;n-1).$$

A simple induction argument shows that the solution to this recurrence is given by

$$S_{-3}(f;n) \quad = \quad P(n)\sum_{k=0}^{n-1} \frac{1}{(4k+1)^2}\,. \tag{39}$$

Next we set $r = -5$ in recurrence (28), and after a similar calculation to the case $r = -3$, arrive at the result

$$S_{-5}(f;n) \quad = \quad P(n)\sum_{j=0}^{n-1} \frac{1}{(4j+1)^2} \sum_{k=0}^{j-1} \frac{1}{(4k+1)^2}\,. \tag{40}$$

The general result, provable by an induction argument, is a multiple series expression for the series $S_{-(2r+1)}(f;n), r = 1,2,3,...$:

$$S_{-(2r+1)}(f;n) \quad = \quad P(n)\left(\sum_{0\le k_1 \le \cdots \le k_r \le n-1} \frac{1}{(4k_1+1)^2 \cdots (4k_r+1)^2}\right). \tag{41}$$

If we let n tend to infinity in (41) we obtain the following result, expressing an r-fold multiple sum in terms of a single summation: for $r = 1,2,3,...$, there holds

$$C_1\left(\sum_{0\le k_1 \le \cdots \le k_r} \frac{1}{(4k_1+1)^2 \cdots (4k_r+1)^2}\right) \quad = \quad \sum_{n=0}^{\infty} \frac{\binom{-\frac{1}{2}}{n}}{(4n+1)^{2r+1}}\,. \tag{42}$$

A more general result, obtained by a similar analysis to the foregoing (see Appendix B, equation (60)), is that for $r = 1,2,3,...$, $m \notin [0, 1/(r+1)]$, and $m \notin \left\{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\right\}$ we have

$$C(m)\left(\sum_{0\le k_1 \le \cdots \le k_r} \frac{1}{(mk_1+1)^2 \cdots (mk_r+1)^2}\right) \quad = \quad \sum_{n=0}^{\infty} \frac{\binom{-\frac{2}{m}}{n}}{(mn+1)^{2r+1}}, \tag{43}$$

where the constant $C(m)$ is given by

$$C(m) \quad = \quad \prod_{k=1}^{\infty}\left(1-\frac{1}{(mk+1)^2}\right). \tag{44}$$

The case $m=-4$ gives a companion result to (42):

$$C_3\left(\sum_{0\le k_1\le\cdots\le k_r}\frac{1}{(4k_1-1)^2\cdots(4k_r-1)^2}\right) \quad = \quad 1-\sum_{n=1}^{\infty}\frac{\binom{\frac{1}{2}}{n+1}}{(4n+3)^{2r+1}},$$

Setting $m=1$ in (43) gives

$$\frac{1}{2}\left(\sum_{1\le k_1\le\cdots\le k_r}\frac{1}{k_1^2\cdots k_r^2}\right) \quad = \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2r}}, \qquad r\in\mathbb{Z}_{\ge 1},$$

a well-known result in the theory of multiple zeta star values. See, for example, [Aoki and Ohno, Theorem 1 with $k=2r$ and $s=r$].

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APPENDIX A

We obtain the well-known expression for the constant C_1 in terms of the gamma function stated earlier in (4) and the continued fraction representation given in (6). The corresponding results for the constant C_3 stated in (5) and (7) can be found in a similar manner.

Gamma function representations Let $u_n = \frac{(n+a_1)(n+a_2)}{(n+b_1)(n+b_2)}$, a rational function of n, where $a_1 + a_2 = b_1 + b_2$. Then the infinite product $\prod_{n=0}^{\infty} u_n$ converges absolutely and has the value

$$\prod_{n=0}^{\infty} \frac{(n+a_1)(n+a_2)}{(n+b_1)(n+b_2)} \quad = \quad \frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(a_1)\,\Gamma(a_2)}\,. \tag{45}$$

The result follows easily from the Weierstrass product formula for the gamma function. For details of the proof and a generalisation of this result see [Whittaker and Watson, p. 238].

By definition

$$\begin{aligned} C_1 &= \prod_{n=1}^{\infty}\left(1 - \frac{1}{(4n+1)^2}\right) = \prod_{n=1}^{\infty} \frac{4n(4n+2)}{(4n+1)^2} \\ &= \prod_{n=1}^{\infty} \frac{n\left(n+\frac{1}{2}\right)}{\left(n+\frac{1}{4}\right)\left(n+\frac{1}{4}\right)} \\ &= \prod_{n=0}^{\infty} \frac{(n+1)\left(n+\frac{3}{2}\right)}{\left(n+\frac{5}{4}\right)\left(n+\frac{5}{4}\right)} \end{aligned}$$

after reindexing. Now apply (45) to find

$$\begin{aligned} C_1 &= \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(1)\Gamma\left(\frac{3}{2}\right)} = \frac{\left(\frac{1}{4}\right)^2\Gamma\left(\frac{1}{4}\right)^2}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{\Gamma\left(\frac{1}{4}\right)^2}{8\sqrt{\pi}}\,, \end{aligned} \tag{46}$$

which is the result stated in (4). The corresponding result for the constant C_3 stated in (5) is found in an exactly similar manner.

Continued fractions There is a general method due to [Stern, p. 266] to convert a product to a continued fraction. We shall apply Stern's method to find a continued fraction representation for the product

$$\prod_{n=0}^{\infty} \frac{(n+a_1)(n+a_2)}{(n+b_1)(n+b_2)}\,, \tag{47}$$

where $a_1 + a_2 = b_1 + b_2$. Results (6), (7) and (8) are particular cases of this general result.

Define the partial products P_n and Q_n of the numerator and denominator of the infinite product (47) by setting $P_0 = 1$, $P_1 = a_1$, and thereafter

$$P_{2n} = \prod_{k=0}^{n-1} (k + a_1)(k + a_2), \quad n \geq 1$$

$$P_{2n+1} = (n + a_1)P_{2n}, \quad n \geq 1,$$

and setting $Q_0 = 1$, $Q_1 = b_1$, and thereafter

$$Q_{2n} = \prod_{k=0}^{n-1} (k + b_1)(k + b_2), \quad n \geq 1$$

$$Q_{2n+1} = (n + b_1)Q_{2n}, \quad n \geq 1.$$

Let $\delta = b_2 - a_1 = a_2 - b_1$. It is not difficult to check that the partial products P_n and Q_n satisfy the following recurrence equations of order 2:

$$P_{2n} = \delta P_{2n-1} + (n - 1 + a_1)\,(n - 1 + b_1)\,P_{2n-2}$$

$$Q_{2n} = \delta Q_{2n-1} + (n - 1 + a_1)\,(n - 1 + b_1)\,Q_{2n-2}$$

and

$$P_{2n+1} = (1 - \delta)P_{2n} + (n - 1 + a_2)\,(n - 1 + b_2)\,P_{2n-1}$$

$$Q_{2n} = (1 - \delta)Q_{2n-1} + (n - 1 + a_2)\,(n - 1 + b_2)\,Q_{2n-2}.$$

By the general theory of continued fractions these recurrences translate to the finite continued fraction expansion

$$\frac{P_{2n}}{Q_{2n}} = 1 + \frac{a_1 - b_1}{b_1 +}\;\frac{a_1 b_1}{\delta +}\;\frac{a_2 b_2}{1 - \delta +}\;\frac{(1 + a_1)\,(1 + b_1)}{\delta +}\;\cdots + \frac{(1 + a_2)\,(1 + b_2)}{1 - \delta \quad +}\;\cdots + \frac{(n - 1 + a_1)\,(n - 1 + b_1)}{\delta}.$$

Letting $n \to \infty$, and remembering (45), yields the infinite continued fraction expansion

$$\frac{\Gamma(b_1)\,\Gamma(b_2)}{\Gamma(a_1)\,\Gamma(a_2)} = 1 + \frac{a_1 - b_1}{b_1 +}\;\frac{a_1 b_1}{\delta +}\;\frac{a_2 b_2}{1 - \delta +}\;\frac{(1 + a_1)\,(1 + b_1)}{\delta +}\;\cdots + \frac{(1 + a_2)\,(1 + b_2)}{1 - \delta \quad +}\;\cdots, \tag{48}$$

where we recall that $a_1 + a_2 = b_1 + b_2$ and $\delta = b_2 - a_1$.

We apply this result to find a continued fraction representation for the constant C_1. As an intermediate step in arriving at (46) we had the following representation for C_1 in terms of the gamma function:

$$C_1 = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(1)\Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(\frac{4}{4})\Gamma\left(\frac{6}{4}\right)}.$$

Applying (48) to this ratio of gamma function values with $a_1 = \frac{4}{4}$, $a_2 = \frac{6}{4}$, $b_1 = b_2 = \frac{5}{4}$ (so that $\delta = b_2 - a_1 = \frac{1}{4}$) we obtain the continued fraction representation

$$C_1 = 1 + \frac{-\frac{1}{4}}{\frac{5}{4}+}\ \frac{\frac{4}{4}\times\frac{5}{4}}{\frac{1}{4}+}\ \frac{\frac{5}{4}\times\frac{6}{4}}{\frac{3}{4}+}\ \frac{\frac{8}{4}\times\frac{9}{4}}{\frac{1}{4}+}\ \frac{\frac{9}{4}\times\frac{10}{4}}{\frac{3}{4}\quad+}\ \cdots.$$

Repeated use of equivalence transformations puts this result into the form

$$C_1 = 1 - \frac{1}{5+}\ \frac{4\times 5}{1+}\ \frac{5\times 6}{3+}\ \frac{8\times 9}{1+}\ \frac{9\times 10}{3\quad+}\ \cdots,$$

whci is the result stated earlier in (6). The continued fraction expansions (7) and (8) for the constants $C_3 = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{2}{4})\Gamma(\frac{4}{4})}$ and $C_1/C_3 = \frac{1}{2}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}$ are found in the same way.

APPENDIX B

We outline how the reults of Section 2.2 may be generalised.

Let α be a complex number. Define a sequence of rational functions $b_n(\alpha, x) \equiv b_n(x)$ in x, for $n = 0, 1, 2, ...$, by putting $b_0(x) = 1$ and setting

$$b_n(x) = \prod_{j=1}^{n} \frac{x-j}{x+j-(2\alpha+1)} \tag{49}$$

for $n \geq 1$. In Section 2.2 above we considered the particular case $\alpha = -\frac{1}{4}$. It is not difficult to verify that the functions $b_n(x), n = 0, 1, 2, ...$, satisfy the following recurrence equation in x:

$$(n-\alpha)^2 b_n(x) = (x-\alpha-1)^2 b_n(x) - (x-1)(x-2\alpha-1)b_n(x-1). \tag{50}$$

Define series $S_r(\alpha, x) \equiv S_r(x)$ by

$$S_r(x) = \sum_{n=0}^{\infty} \binom{2\alpha}{n} (\alpha-n)^r b_n(x). \tag{51}$$

We will be interested in the case where r is negative, so from now on we suppose that $\alpha \neq 0, 1, 2, \ldots$. Note that if N is a positive integer then the series $S_r(N)$ terminates since $b_k(N) = 0$ for $k \geq N$.

Let $N \geq 2$ be an integer. If we set $x = N$ in (50), then multiply the resulting equation by the factor $\binom{2\alpha}{n}(\alpha - n)^r$ and sum over n, we see that the terminating series $S_r(N)$ satisfies the recurrence equation

$$S_{r+2}(N) \quad = \quad (N - \alpha - 1)^2 S_r(N) - (N-1)(N - 2\alpha - 1) S_r(N-1). \tag{52}$$

We will use this recurrence to investigate the odd -indexed series $S_{2r+1}(N)$ when r is a nonpositive integer. The initial case is when $r = 0$. Using the method of telescopic summation it is not difficult to show that

$$S_1(N) \quad = \quad 0 \quad \text{for } N = 2, 3, 4, \ldots . \tag{53}$$

Now put $r = -1$ in the recurrence (52). Then by (53) we have for $N = 2, 3, 4, ...,$

$$0 \quad = \quad (N - \alpha - 1)^2 S_{-1}(N) - (N-1)(N - 2\alpha - 1) S_{-1}(N-1), \tag{54}$$

from which we get

$$\begin{aligned} S_{-1}(N) \quad &= \quad \left(1 - \frac{\alpha^2}{(N - \alpha - 1)^2}\right) S_{-1}(N-1) \\ &= \quad \left(1 - \frac{\alpha^2}{((N-1) - \alpha)^2}\right) \cdots \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right) S_{-1}(1). \end{aligned}$$

From the definition of the series $S_r(x)$ in (51) we see that $S_{-1}(1) = \frac{1}{\alpha}$. Thus we have established the identity

$$\alpha S_{-1}(N) = \sum_{n=0}^{\infty} \binom{2\alpha}{n} \frac{\alpha}{\alpha - n} b_n(N) = \prod_{k=1}^{N-1} \left(1 - \frac{\alpha^2}{(k - \alpha)^2}\right), \tag{55}$$

where N is a positive integer and $\alpha \in \mathbb{C} - \{0, 1, 2, ...\}$.

Now we further restrict α by requiring that $\mathrm{Re}(\alpha) > -1$, so that the hypergeometric series $\displaystyle\sum_{n=0}^{\infty} \binom{2\alpha}{n} \frac{\alpha}{\alpha - n}$ converges as noted in Theorem 1. Finally, let $N \to \infty$ in (55). With a little care [2] we can justify taking the

[2] As previously noted, an example of the type of reasoning needed here can be found in Knopp's Theory and Application of Infinite Series, Dover Publ. 1990, §23, p. 193.

termwise limit in the sum on the left-hand side of (55) to arrive at the sum-to-product identity

$$\sum_{n=0}^{\infty}\binom{2\alpha}{n}\frac{\alpha}{\alpha-n}=\prod_{k=1}^{\infty}\left(1-\frac{\alpha^2}{(\alpha-k)^2}\right),\quad \mathrm{Re}(\alpha)>-1,\alpha\neq 0,1,2,\dots. \tag{56}$$

Thus we have arrived at part (i) of Theorem 1 from Section 2.2 by a different approach.

The infinite product on the right-hand side of (56) converges absolutely and defines a meromorphic function in the complex variable α, with simple poles when $\alpha = 1,2,3,\dots$. From the series representation we see that the residue of this function at the pole $\alpha = n$ equals $n\binom{2n}{n}$. We can view the infinite product as the analytic continuation of the series on the left-hand side of (56) to the complex plane. Next we generalise the results of Section 2.3.

The series $S_{-3}(N), S_{-5}(N)$,... Having found a product formula for the series $S_{-1}(N)$ when N is a positive integer in (55), we can use recurrence (52) to succesively obtain results for the terminating series $S_{-3}(N)$, $S_{-5}(N)$ and so on. After a short calculation we find

$$S_{-3}(N) = \frac{1}{\alpha}\prod_{k=1}^{N-1}\left(1-\frac{\alpha^2}{(k-\alpha)^2}\right)\sum_{k=0}^{N-1}\frac{1}{(k-\alpha)^2}, \tag{57}$$

$$S_{-5}(N) = \frac{1}{\alpha}\prod_{k=1}^{N-1}\left(1-\frac{\alpha^2}{(k-\alpha)^2}\right)\sum_{0\le k_1\le k_2\le N-1}\frac{1}{(k_2-\alpha)^2(k_1-\alpha)^2}, \tag{58}$$

and in general

$$S_{-(2p+1)}(N) = \frac{1}{\alpha}\prod_{k=1}^{N-1}\left(1-\frac{\alpha^2}{(k-\alpha)^2}\right)\sum_{0\le k_1\le\cdots\le k_p\le N-1}\frac{1}{(k_p-\alpha)^2\cdots(k_1-\alpha)^2}. \tag{59}$$

Letting $N\to\infty$ in (59) yields

$$\sum_{n=0}^{\infty}\binom{2\alpha}{n}\frac{1}{(\alpha-n)^{2p+1}} = \frac{1}{\alpha}\prod_{k=1}^{\infty}\left(1-\frac{\alpha^2}{(k-\alpha)^2}\right)\sum_{0\le k_1\le\cdots\le k_p}\frac{1}{(k_p-\alpha)^2\cdots(k_1-\alpha)^2}, \tag{60}$$

where for convergence of the series on the left-hand side we require $\mathrm{Re}(\alpha) > -(p+1)$. As an example, when $\alpha = -\frac{1}{2}$ the result reads

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^{2p+1}} = \frac{\pi}{4}\sum_{0\le k_1\le\cdots\le k_p}\frac{1}{(2k_p+1)^2\cdots(2k_1+1)^2}. \tag{61}$$

This result was also given in notes I recently upoaded to A245244 (see equation 29).