The Multiply Perfect Numbers Page

Introduction

Let *σ(n)* be the number theoretic function which denotes the sum of all divisors of a natural number *n*. If *σ(n)* is an integral multiply of *n*, then *n* is denoted as a **multiply perfect number** or **k-fold perfect number** (also called **multiperfect number** or **pluperfect number**). Call *σ(n)/n* **abundancy** (also called **index** or **multiplicity**) of *n*. A multiply perfect number is called **proper** if its abundancy is > 2. For example consider the divisors of the number *120*:
*1+2+3+4+5+6+8+10+12+15+20+24+30+40+60+120=σ(120)=σ(2^3*3*5)=σ(2^3)*σ(3)*σ(5)=(1+2+4+8)*(1+3)*(1+5)=15*4*6=360=3*120*.
Hence 120 is a 3-fold perfect number.

Status

Abundancy	Count	When last number was discovered	Which was last?	Are all discovered?	Estimated total number
1	1	-	-	yes and proved	1
2	50	2017-12-26	18.4889706	no, there are infinitely many	∞
3	6	<= 1643	3.2049844	yes	6
4	36	<= 1929	4.3351682	yes	36
5	65	<= 1990	5.1744360	yes	65
6	245	1993-05-??	5.6720844	yes	245
7	516	1994-01-09	5.9403364	almost surely yes	~ 515
8	1135	2017-05-20	6.3396518	probably yes	~ 1140
9	2095	2013-01-10	7.2802453	no	~ 2200
10	1164	2013-01-03	7.2933919	no	~ 4500
11	1	2001-03-13	8.3870050	no	~ 10000

In column "Which was last?" the identifier *ln(ln(MPN))* is given for those which were verfied by me. I checked these numbers only for those MPNs reported before 2017-05-23.
We have a total of 5311+3 (of which 5263 have an abundancy > 2) known and claimed MPNs until 2018-01-07.
It is extremly probable, that all proper MPNs with abundancy <= 7 are discovered.

Data

Richard Schroeppel's archive of 2094 MPNs built 1995-12-13 .
The collection of 5311 MPNs from 2014-01-01(gziped to 918 kB) sorted by abundancy and magnitude. It is grown out of Rich's database --- thanks ---, and transformed into a new format, such that each multiply perfect number allocates one line with all its additional informations in the form:
`M|ln_ln|rich_id,deep|dpf,tpf|date|name|number|comment`

- `|` is a separator character between the fields and is not allowed inside any field. Except of the last field, `comment`, all other fields are obligatory, but they can be empty, e.g. if the discovery date or person is unknown.
- `M` indicates the abundancy of the number as a lower case letter, such that the letters a,b,c,d,... correspond to the abundancies 1,2,3,4, ...
- `ln_ln` gives the decimal value of $\log_e \log_e$ of the number and is rounded to 7 decimal places after the period. This serves now as a unique identifier to each number, also.
- `rich_id` is Rich's unique identifier for this number which encodes the abundancy appended by the exponents of at most the three primes 2,3,5. These exponents are encoded alternatingly in a 26-base system made up of the letters `a-z` und a 10-base system made up of the digits `0-9`. The letter `@` is used if a prime exponent is zero. In the case that this identifier is still ambigious, it is appended by a lower case letter which serves as a counter.
  R. Sorli likes to avoid this further counter and recommends to use not only the primes 2,3,5, but as less as further necessary for unequivocality using Rich's scheme of alternatingly letters and digits for encoding the corresponding exponents (at most up to and including 23 is sufficient until now).
- `deep` indicates the minimal number of successive primes (starting with 2) whose exponents must be given to reconstruct the MPN straight forward (without knowing its abundancy).
- `dpf` indicates the number of different prime factors.
- `tpf` indicates the number of total prime factors.
- `date` gives the year of the first (or independent) discovery in the form `YYYY-MM-DD` as long as month and day (and year) is known.
- `name` gives the name of the discoverer. Like in the date field, multiple independent discoveries are separated by commas.
- `number` is given in its prime factorization of the form:`base1^expo1.base2^expo2.base3^expo3. ....` where the `basei` indicates a prime of an ascending list of primes and the `expoi` indicates the corresponding nonzero exponent; an exponent value of 1 is omitted together with its leading `^`. In the case of the larger 2-perfect numbers, the odd prime (`2^??-1`) in the factorization is given as `M??` to save space with `??` giving the **Mersenne Prime exponent** as decimal number.

Search this database of 5311 MPNs
Each search-pattern matches literally, except the wild-characters *,?,!,$ which mean:
* -- arbitrary, maybe empty, sequence of characters
? -- exactly one character
! -- begin of record
$ -- end of record.
Hint: orient on the field separator | or the comma in each record to get easier what you want.

Pattern [ ] to search.
output fields of records 1 - 9
Clear Look up

And a further list of multiply perfect numbers sorted only by their factorizations (built from the 5311 MPNs of the master list and gziped 200 kB). It has each line cut down to 120 columns and no discovery information or comments are given. Rich's out-dated indentifier is omitted, but a $\log_{10}$(MPN) is given for traditional reasons.

Finally, here are the (email-extracted and edited) 2 claimed new (since the database update) found MPNs which I irregularly append (last change: 2017-05-23). To verify these numbers, three steps must be taken: 1) verify for each number n given in its prime factorization that all factors are really prime numbers, 2) compute $o(n)$ which envolves the factorization of large numbers and then check n's claimed abundancy 3) and lastly test whether n is really new.

Algebraic Transformations

Let $o(n*X)=k*n*X$ and $o(n*Y)=k*n*Y$, with $gcd(n,X)=1=gcd(n,Y)$, i.e. they have no primes in common. Then yields: if $n*X$ is a k-perfect number then $n*Y$ is one, too. Such a pair (X,Y) is denoted as a **substitution**. To avoid trivial substitutions, one demands further that prime-powers whose prime occuring in both components have different exponents. Call the **size** of a substitution the number of different primes involved. Below are some components listed for pairs `(A,B), (C,D), (E,F), (G,H), (I,J), (K,L), (M,N), (P,Q), (R,S), (T,U)` and `(V,W)`:

```
A = 19^2.127^1.151^0.911^0
B = 19^4.127^0.151^1.911^1
size(A,B) = 4 , k = 4,5,6,7,8,9
[occurs 257 times for the known MPNs]

C = 23^1.37^3.73^1.79^0.137^2
D = 23^2.37^1.73^0.79^1.137^1
size(C,D) = 5 , k = 6,7,8,9
[occurs 21 times for the known MPNs]

E = 13^1.31^2.61^1.83^1.97^0.331^1
F = 13^2.31^1.61^2.83^0.97^1.331^0
size(E,F) = 6 , k = 6,7,8
[occurs 31 times for the known MPNs]

G = 3^10.107^1.137^0.547^0.1093^0.3851^1
H = 3^6.107^0.137^1.547^1.1093^1.3851^0
size(G,H) = 6 , k = 4,5,6
[occurs 31 times for the known MPNs]

I = 5^0
J = 5^1
size(I,J) = 1 , k = 5
[occurs 20 times for the known MPNs]

K = 7^5.17^1.37^1.43^1.67^0.307^0.1063^0
L = 7^8.17^2.37^2.43^0.67^1.307^1.1063^1
size(K,L) = 7 , k = 7
[occurs 9 times for the known MPNs]

M = 7^4.13^4.31^1.43^1.61^1.79^0.97^0.157^0.631^0.30941^1
N = 7^9.13^5.31^0.43^2.61^2.79^2.97^1.157^1.631^1.30941^0
size(M,N) = 10 , k = 5,6,7,8
[occurs 4 times for the known MPNs]

P = 2^38.53^1.59^0.157^0.229^1.8191^1.43331^0.121369^1.3033169^0.715827883^0.2147483647^0
Q = 2^61.53^0.59^1.157^1.229^0.8191^0.43331^1.121369^0.3033169^1.715827883^1.2147483647^1
size(P,Q) = 11 , k = 5,6
[occurs 9 times for the known MPNs]

R = 13^14.139^0.157^1.181^1.191^2.199^1.229^1.397^1.827^0.1163^1.4651^1.8269^0.14197^0.28393^0.30941^1.40493^1.161971^1
S = 13^11.139^1.157^2.181^2.191^1.199^0.229^2.397^0.827^1.1163^0.4651^0.8269^1.14197^1.28393^1.30941^0.40493^0.161971^0
size(R,S) = 17 , k = 9
[occurs 1 time for the known MPNs]

T = 3^4.11^3.13^1
U = 3^5.11^1.13^2
size(T,U) = 3 , k = 6
[occurs 8 times for the known MPNs]

V = 5^2.7^2.11^1.31^1.71^0
W = 5^4.7^3.11^2.31^0.71^1
size(V,W) = 5 , k = 6
[occurs 8 times for the known MPNs]
```

They are as more important, as fewer primes are involved in such a substitution. As larger the smallest prime in such a pair is, as more usefull it is, too. On the other hand, as smaller the quotient of the number of different primes in n to the size of the substitution becomes, as more worthless such a substitution becomes for this n.

Historical Development of known MPNs

period / year	Antiquity	Middle Ages	< 1910	< 1915	< 1930	< 1955	< 1980	< 1990	1991	1992	1993	1994	1995
known MPNs	5	35	47	235	347	556	596	611	713	1179	1821	1983	2101
increament	5	30	12	188	112	209	40	15	102	466	642	162	118

period	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
known MPNs	2335	3051	3259	3464	4501	4997	5040	5050	5147	5189	5222	5245	5271	5287	5301	5303	5307	5311
increament	234	716	208	205	1037	496	43	10	97	42	33	23	26	16	14	2	4	4

The sign < in the row **period** means "up to and including" the given year.
The value in the row **increament** indicates the new known MPNs compared to the previous time-period.

Now a list of the discoverer names follows according to Rich Schroeppel (report him, if you think you aren't credited sufficiently):

Discoverer	count	period	Discoverer	count	period
Armengaud&Woltman&et.al.	1*	1996	Brown	78	1954
Cameron&Woltman&Kurowski&et.al.	1*	2001	Carmichael	98	1907-1911
Cataldi	2*	1588	Clarkson&Woltman&Kurowski&et.al.	1*	1998

codex-Lat.-Monac.-	1*	1456	Colquitt&Welsh	1*	1988
Cooper&Boone&Woltman&Kurowski&et.al.	2*	2005-2006	Cooper&Woltman&Kurowski&et.al.	1*	2013
Cunningham	3	1902	Descartes	9	1638-1639
Dokshitzer&Vasyunin	1	1993	Euclid	4*	-275
Euler	1*	1772	Elvenich&Woltman&Kurowski&et.al.	1*	2008
Fermat	11?	1636-1643	Findley&Woltman&Kurowski&et.al.	1*	2004
Flammenkamp	137	1993-2000	Flammenkamp&Woltman	344	2000-2001
Franqui&Garcia	135	1953-1954	Freniclei	3	1638
Gage&Slowinski	3*	1992-1996	Gillies	3*	1963
Gretton	123	1990-1993	Gretton&Schroeppel	5	1991-1992
Hajratwala&Woltman&Kurowski&et.al.	1*	1999	Harrison&Moxham	3	2000
Helenius	1261	1992-1997	Helenius&Schroeppel	16	1993
Hurwitz&Selfridge	2*	1961	*in-Mersenne-epoch*	4	1639-1643
Jumeau	1	1638	Kapek&Moxham	1	2000
Lehmer	5	1901	Lucas	1*	1876
Martinson&Moxham	1	2000	Mason	89	1911
Moxham	1006	1995-2001	Nelson&Slowinski	1*	1979
Nickel&Noll	1*	1978	Noll	1*	1979
Nowak&Woltman&Kurowski&et.al.	1*	2005	Perrier&Woltman	38	2000
Pervouchine	1*	1883	Poulet	145	1929,1954
Powers	2*	1911-1914	Pythagoras	1*	-500
Recorde	1	1557	Riesel	1*	1957
Roberts&Moxham	2	1997	Robinson	5*	1952
Schroeppel	7	1983-1991	Shafer&Woltman&Kurowski&et.al.	1*	2003
Slowinski	3*	1982-1985	Smith&Woltman&Kurowski&et.al.	1*	2008
Sorli&Moxham	27	1997-1998	Sorli&Woltman	9	2000-2001
Spence&Woltman&et.al.	1*	1997	Strindmo&Woltman&Kurowski&et.al.	1*	2009
Tuckerman	1*	1971	Whiteside&Moxham	2	1999
Woltman	1799	1997-2013	Yoshitake	17	1974-1993

The column 'count' counts the discoveries until 2013-12-31 and an asterik * indicates that this person has "only" discovered 2-perfect number(s).

Abundancy	First discovered MPN	Is it smallest?	Date	Discoverer
1	1	yes	ancient	-
2	6	yes	ancient	-
3	120	yes	ancient	-
4	30240	yes	~ 1638	R. Descartes
5	14182439040	yes	~ 1638	R. Descartes
6	34111227434420791224041472000	no	1643	P. Fermat
7	6.9545266398342727... *10^70	no	1902	A. J. C. Cunningham
8	2.34111439263306338... *10^161	no	1929	P. Poulet
9	7.9842491755534198... *10^465	no	1992-04-15	F. W. Helenius
10	2.86879876441793479... *10^923	no	1997-05-13	R. M. Sorli
11	2.51850413483992918... *10^1906	no	2001-03-13	G. F. Woltman

Various Records

The factorizations of smallest -- for k >= 9 "only" smallest known -- k-perfect number for fixed abundancy are:

- Abundancy 1:
  1
- Abundancy 2:
  2 3
- Abundancy 3:
  2^3 3 5
- Abundancy 4:
  2^5 3^3 5 7
- Abundancy 5:
  2^7 3^4 5 7 11^2 17 19
- Abundancy 6:
  2^15 3^5 5^2 7^2 11 13 17 19 31 43 257
- Abundancy 7:
  2^32 3^11 5^4 7^5 11^2 13^2 17 19^3 23 31 37 43 61 71 73 89 181 2141 599479
- Abundancy 8:
  2^62 3^15 5^9 7^7 11^3 13^3 17^2 19 23 29 31^2 37 41 43 53 61^2 71^2 73 83 89 97^2 127 193 283 307 317 331 337 487 521^2 601 1201 1279 2557 3169 5113

92737 649657

- Abundancy 9:
  2^104 3^43 5^9 7^12 11^6 13^4 17 19^4 23^2 29 31^4 37^3 41^2 43^2 47^2 53 59 61 67 71^3 73 79^2 83 89 97 103^2 107 127 131^2 137^2 151^2 191 211 241 331 ...
- Abundancy 10:
  2^175 3^69 5^29 7^18 11^19 13^8 17^9 19^7 23^9 29^3 31^8 37^2 41^4 43^4 47^4 53^3 59 61^5 67^4 71^4 73^2 79 83 89 97 101^3 103^2 107^2 109 113 127^2 ...
- Abundancy 11:
  2^468 3^140 5^66 7^49 11^40 13^31 17^11 19^12 23^9 29^7 31^11 37^8 41^5 43^3 47^3 53^4 59^3 61^2 67^4 71^4 73^3 79 83^2 89 97^4 101^4 103^3 109^3 ...

Abund	Smallest value	Largest value	Lowest 2-power	Highest 2-power	Largest eff. expo.	Fewest factors *	Most factors *
2	6 (0.5831981)	*10^34850339.2 (18.2006059)*	1 (0.5831981)	*57885160 (18.2006059)*	0 (0.5831981)	2 (0.5831981)	*57885161 (18.2006059)*
3	120 (1.5660066)	10^10.469 (3.2049844)	3 (1.5660066)	14 (3.2049844)	7 (3.2049844)	5 (1.5660066)	19 (3.2049844)
4	30240 (2.3337853)	10^45.204 (4.6452114)	2 (2.6806218)	37 (4.5312242)	17 (4.5312242)	8 (2.3415138)	57 (4.6365489)
5	10^10.148 (3.1516786)	10^100.276 (5.4419551)	7 (3.1516786)	61 (5.0442101)	45 (5.1777077)	17 (3.1516786)	103 (5.3046236)
6	10^20.189 (3.8391453)	10^192.162 (6.0923690)	15 (4.0050961)	92 (6.0923690)	79 (6.0362173)	31 (3.8391453)	164 (6.0923690)
7	10^56.150 (4.8620623)	10^312.074 (6.5772722)	29 (5.2987825)	177 (6.5408889)	108 (6.4271649)	71 (4.8620623)	307 (6.5528603)
8	10^132.917 (5.7237604)	*10^613.291 (7.2528723)*	47 (5.7578915)	*253 (7.1699823)*	156 (7.2528723)	137 (5.7237604)	*466 (7.1485402)*
9	*10^286.749 (6.4926404)*	*10^1165.883 (7.8952666)*	*99 (6.6259527)*	*380 (7.8522986)*	*283 (7.7478296)*	*257 (6.4926404)*	*747 (7.8522986)*
10	*10^638.652 (7.2933919)*	*10^1877.645 (8.3718042)*	*175 (7.2933919)*	*534 (8.3718042)*	*434 (8.2500454)*	*492 (7.2933919)*	*1172 (8.3718042)*
11	*10^1906.401 (8.3870050)*	*10^1906.401 (8.3870050)*	*468 (8.3870050)*	*468 (8.3870050)*	*469 (8.3870050)*	*1139 (8.3870050)*	*1139 (8.3870050)*

Italic printed values perhaps change in the future.
The identifiers for each MPN are given in parenthesis.
(*) The word factors means prime-factors.

Discovery dates of proper Multiply Perfect Numbers

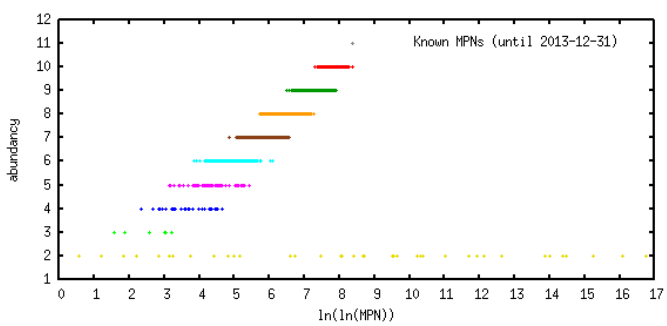
Abund	First MPN	Smallest MPN	Largest MPN	Latest MPN
3	ancient (1.5660066)	ancient (1.5660066)	1643 (3.2049844)	1643 (3.2049844)
4	1638 (2.3337853)	1638 (2.3337853)	1911 (4.6452114)	1929 (4.3351682)
5	1638 (3.1516786)	1638 (3.1516786)	1911 (5.4419551)	1990 (5.1744360)
6	1643 (4.1850902)	1907 (3.8391453)	1992 (6.0923690)	1993-05-?? (5.6720844)
7	1902 (5.0944883)	1911 (4.8620623)	1993 (6.5772722)	1994-01-09 (5.9403364)
8	1929 (5.9177287)	1990 (5.7237604)	*1996-04-23 (7.2528723)*	*2017-05-20 (6.3396518)*
9	1992-04-15 (6.9780083)	*1995-12-06 (6.4926404)*	*2001-01-11 (7.8952666)*	*2013-01-10 (7.2802453)*
10	1997-05-13 (7.6621574)	*2013-01-03 (7.2933919)*	*2001-01-28 (8.3718042)*	*2013-01-03 (7.2933919)*
11	2001-03-13 (8.3870050)	*2001-03-13 (8.3870050)*	*2001-03-13 (8.3870050)*	*2001-03-13 (8.3870050)*

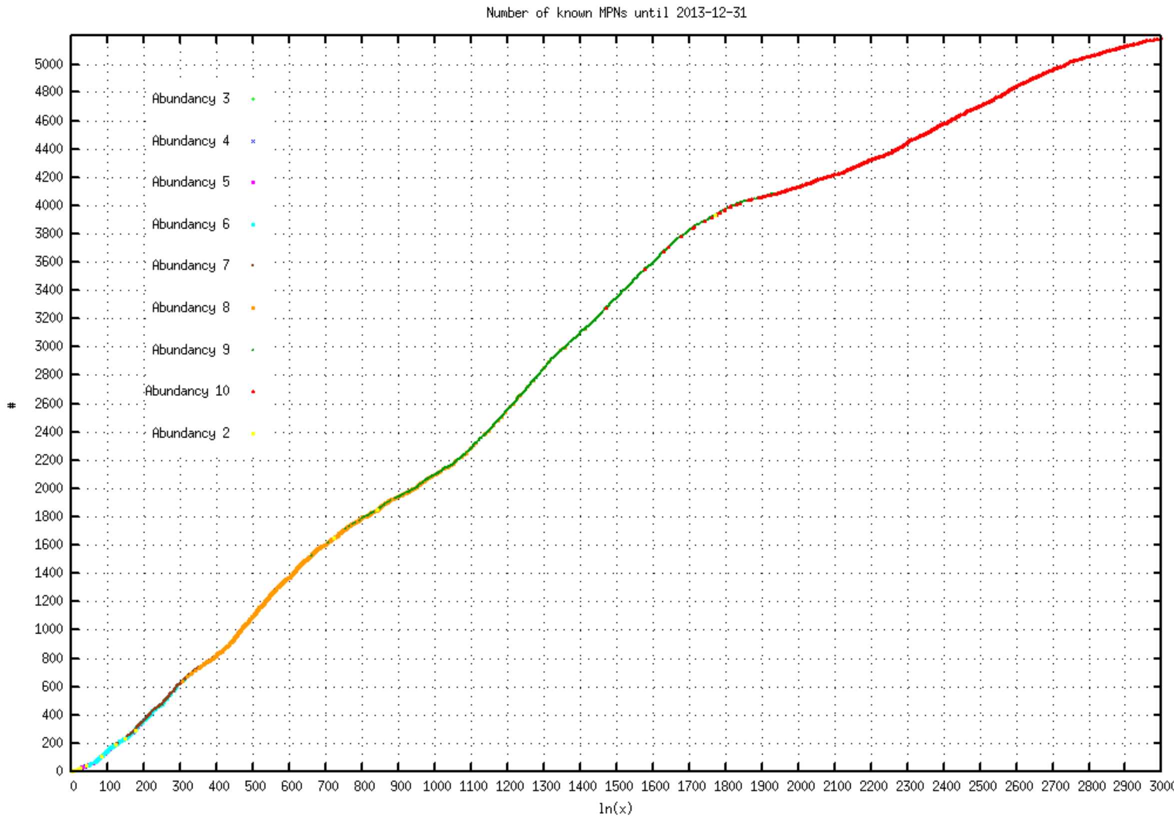
Italic printed values perhaps change in the future.

Old conjectures

- The only odd multiply perfect number is 1. Consequently each (2-fold) perfect number is even (claimed already in the Middle Ages).
- There are infinitely many perfect, i.e. 2-fold multiply perfect, numbers.
- For each fixed abundancy > 2, there are only finitely many multiply perfect numbers.
- For each fixed prime power, there is at least one MPN which has exactly this prime power in its prime factorization (lowest two-power-exponents for to-discover MPNs are 331, 335, 336, ...).

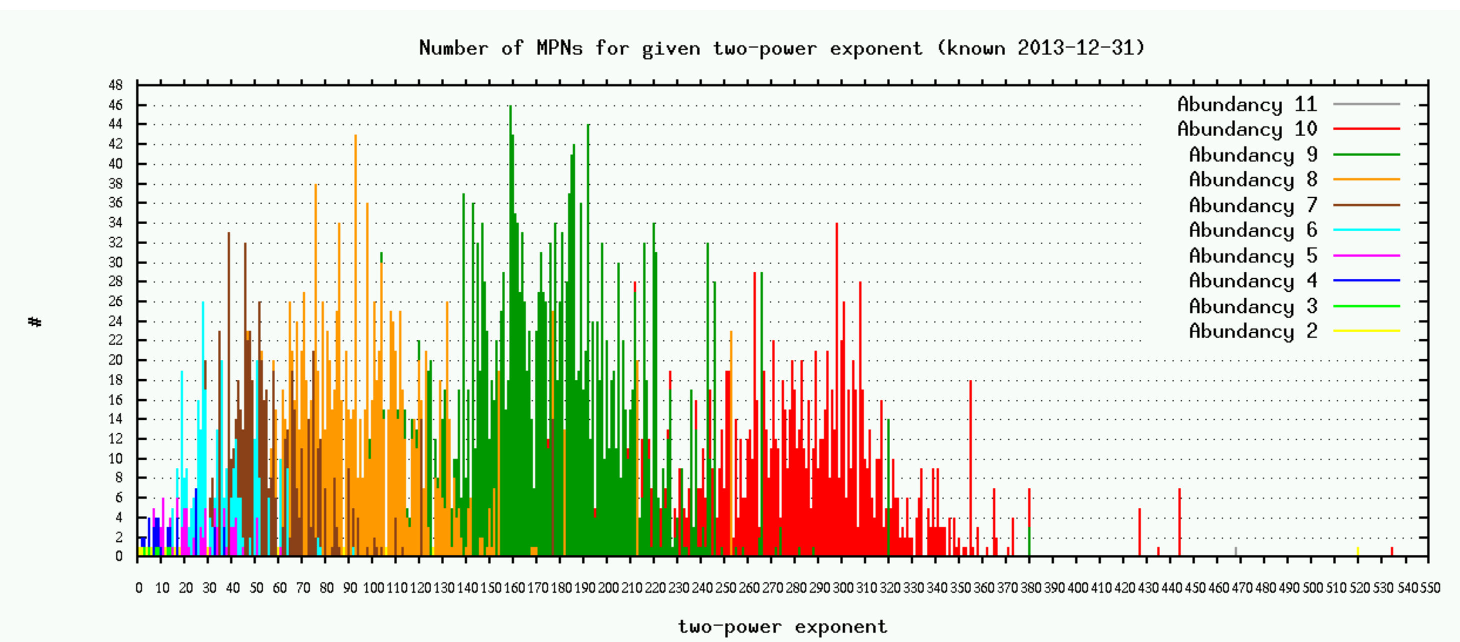
Distribution of the 5311 (until 2013-12-31) known Multiply Perfect Numbers





If the number of MPNs up to a given limit x is proportional to ln(x), then the density of the MPNs should be proportional to 1/x. And a further picture to visualize the density of the known MPNs. In first approximation it may be a Poisson-distribution. You guess that surely there are missing MPNs at last at the currently known 4000th MPN, but also highly probable earlier. Finally a best linear fit for the smallest 3600 known MPNs except the first 100 MPNs to avoid 'start-effects' was done on a logarithmic scale. For these 3500 numbers this least-quadratic error approximation results a correlation coefficient of 0.99815 with *# MPN ~ 2.2328ln(x)-48.3655*. Richard Schroeppel stated that he constructed all MPNs < 10^{70} in the eighties of the twenty-century (but probably only 1990), i.e. he proved that there exist exactly 258 MPNs smaller than 10^{70} -- and besides showed that there are no odd MPNs < 10^{90} except 1. NEW In March 2008 Achim Flammenkamp finally constructed all MPNs < e^{350} by an exhaustive tree-search, i.e. proved that there are no further but the 730 known MPNs. Here is a small paper presenting the theoretical and practical background to this computation.

The Two-Power exponent of a MPN



And here is a false color diagram for the distribution of these MPNs

Let us introduce Fred Helenius' notion of the **effective exponent**. The prime $p=2$ is special compared to other primes, because $p-1$ has only the trivial divisor 1. A consequence of this fact is, only if p equals 2, then numbers of the form p^n-1 could be prime. Assume a fixed MPN m has in its prime factorization the two-power exponent k. This means that $o(m)$ has a factor $2^{k+1}-1$. Depending on k this factor may have some prime divisors of the form 2^n-1. Because we are considering a MPN, $o(2^n-1) = 2^n$ must be also a factor of this MPN itself, if 2^n-1 occurs exactly onetime! Hence the original prime-power 2^k may produce immediately further two-powers. Or put it in other words: only the `remainder' of this exponent k must be produced by other prime-powers. So, the exponent is effectively reduced in consideration to the to-be-generated two-

power factors by other prime-powers. Primes of the form 2^n-1 are called Mersenne Primes and are known up to high values of n. Thus we have to check which Mersenne Prime 2^n-1 divide a given $2^{k+1}-1$ exactly onetimes. Subtracting $k+1$ by such Mersenne Prime exponents n gives the **effective exponent of *k***. There is a small blemish in this model: small primes of the form $q=2^n-1$ may as well be produced by other prime-powers (than two-powers) of a MPN, such that the exponent of q is not 1 in the factorization of the MPN, but larger. Hence we have probably overestimated the correction of the two-power exponent a bit.
The two-power exponent k of a (2-fold) perfect number has always an effective exponent of 0. For such an effective exponent of 0, there seems to exist only the corresponding (2-fold) MPN except in the case k equals 2. If $k = in-1$ with n the exponent of a Mersenne Prime and i a small number, the effective exponent is at most $n(i-1)$, roughly $1-1/i$ of the given exponent k. Such exponents k are typically the largest two-power exponents for which a MPN for a fixed proper abundancy exists: $61=2*31-1$, $92=3*31-1$, $177=2*89-1$, $253=2*127-1$, $320=3*107-1$, $380=3*127-1$.
Finally, for a given two-power exponent k the effort to compute a MPN with an even factor of 2^k seems more related to its effective exponent than to k --- this is heuristically convincing and seems likely to be the reason this quantity was invented --- and lastly the distribution of the prime-exponents of a `typical' MPN seems proportional to the effective exponent than to the two-power exponent.

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- Alan L. Brown, Multiperfect Numbers - Cousins of the Perfect Numbers - No. 1, Recreational Mathematics Magazine #14, Jan/Feb 1964.
- Motoji Yoshitake, Abundant Numbers, Sum of Whose Divisors are an Integer Times the Number, Sugaku Seminar, v.18 n.3 p50-55, 1979.
- private communications from M. Garcia, Stephen Gretton, M. Yoshitake, Fred Helenius, and Achim Flammenkamp.

Further Infos

- Fred Helenius's Multiperfect numbers
- Ron Sorli's Thesis: *Algorithms in the Study of Multiperfect and Odd Perfect Numbers* 2003, Sydney, Australia and its first 9 pages .
- A gziped-tar archive containing the sigma-chain database and its C-sourcecode library (486kB) to access and maintain it (NEW version 4.1d).
- Eric Weisstein's explanations of MPNs. .
- The haunt for improper Multiperfect Numbers :)

Please sent **any comments or questions** concerning this web page to:

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