Golay-Littlewood Problem

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Two independent streams of investigation, one from digital communications engineering and the other from complex analysis on the unit circle, come together in this essay [1, 2, 3, 4, 5].

0.1. Merit Factor of Binary Sequences. Given a sequence \( a_0, a_1, a_2, \ldots, a_n \) where each \( a_j = \pm 1 \), define the \( k^{th} \) **acyclic autocorrelation** to be

\[
c_k = \sum_{j=0}^{n-k} a_j a_{j+k} \quad \text{for } 0 \leq k \leq n; \quad c_k = c_{-k} \quad \text{for } -n \leq k < 0
\]

and the **merit factor** to be the ratio

\[
F = \frac{c_0^2}{\sum_{k \neq 0} c_k^2} = \frac{(n + 1)^2}{2 \sum_{k=1}^{n} c_k^2}.
\]

Identifying binary sequences \( \{a_j\} \) whose autocorrelations \( \{c_k\} \) are jointly as small as possible, for fixed \( n \), is important for engineering design purposes. The “best” sequences are those with the largest merit factor \( F \). As an example, the sequence 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1 has the largest \( F \) value 169/12 = 14.0833... among all such with \( n = 12 \). As another example, the sequence 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0 has the largest \( F \) value 121/10 = 12.1 among all such with \( n = 10 \). No other merit factor exceeding 10 is known for any \( n \); a proof that 169/12 and 121/10 are the maximum possible values for \( F \) is still open.

0.2. \( L_4 \) Norm of Polynomials on Unit Circle. Given a polynomial of complex variable \( z \):

\[
f(z) = \sum_{j=0}^{n} a_j z^j
\]

the \( L_p \) norm of \( f \) over the unit circle for \( p \geq 1 \) is

\[
\|f\|_p = \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right]^{1/p}.
\]

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Since the complex conjugate $\bar{z}$ is equal to $1/z$ and all polynomial coefficients $a_j$ are real, we have $f(z) = f(\bar{z}) = f(1/z)$. Hence
\[ |f(z)|^2 = f(z) f\left(\frac{1}{z}\right) = c_0 + \sum_{k \neq 0} c_k z^k \]
and, after integrating, $\|f\|_2^2 = c_0 = n + 1$ because each $a_j = \pm 1$. Also, we have
\[ |f(z)|^4 = f(z)^2 f\left(\frac{1}{z}\right)^2 = \sum_k c_k^2 + \sum_{k+\ell \neq 0} c_k c_\ell z^{k+\ell} \]
and, after integrating, $\|f\|_4^4 = \sum c_k^2 = (n + 1)^2(1 + 1/F)$. Thus Littlewood’s question [6, 7] about how closely the ratio $\|f\|_4^4 / \|f\|_2^2$ can approach 1 as $n \to \infty$ translates into Golay’s question [8, 9, 10, 11, 12, 13] about the limit supremum of $F$.

### 0.3. Bounds on Asymptotic Behavior

On the one hand, let $\xi = 1.157677...$ denote the smallest zero of $27x^3 - 498x^2 + 1164x - 722$. Jedwab, Katz & Schmidt [14] proved that there is a Littlewood polynomial sequence $\{f_n\}$ such that $\deg(f_n) \to \infty$ and
\[ \frac{\|f_n\|_4^4}{\|f_n\|_2^2} \to 4\sqrt{\xi} = 1.037282... \]
as $n \to \infty$. As a consequence,
\[ \limsup_{n \to \infty} F_n \geq \eta = \frac{1}{\xi - 1} = 6.342061.... \]
The preceding best result, namely $\xi = 7/6 = 1.16...$ ($\eta = 6$), had remained in place for more than twenty years [15, 16]. Recent numerical computations indicate that $\xi = 1.1553...$ ($\eta = 6.4382...$) is feasible. We might have to wait a long time for rigorous verification of this result because, in the words of [17], “inclusion of the steep descent algorithm ... would seem to make a proof much more difficult”. Theory lags considerably behind experiment here: there is good evidence that $\eta > 8$ or even $\eta > 8.5$. Merit factors exceeding 9 are not uncommon for sequence lengths $\approx 100$, but it is difficult to project whether such extremities will continue to grow slowly or level off [18, 19].

On the other hand, no one has proved that the limit supremum of $F$ is necessarily finite. (An argument in [11, 20] that it is approximately 12.32 is only heuristic.) This would be good to see someday.

Imagine the set of all sequences of length $n + 1$, endowed with the uniform distribution. Draw one such sequence and compute $F$. The mean value of $1/F$ is exactly
[21, 22]

\[ E \left( \frac{1}{F} \right) = \frac{n}{n+1} \to 1 \]
as \( n \to \infty \). An exact expression for \( \text{Var}(1/F) \) is not available, but it is \( O(1/n) \) according to [4]. Thus most sequences should have merit factor close to 1 [23]. What else can be said about the distribution of \( 1/F \) or, indeed, of \( F \) itself?


0.4. Addendum. Choi [30] supplemented the result \( E (\|f\|_4^4) = (n + 1)(2n + 1) \) with a new one:

\[ \text{Var} (\|f\|_4^4) = \frac{8}{3} (n + 1) (2n^2 - 2n + 3) - 8 \left[ \frac{n^2 + 2n + 2}{2} \right] \]
giving a formula for \( \text{Var}(1/F) \) as a corollary. Golay’s constant is, to higher precision,

\[ 12.3247958363... = \frac{2y^2}{2y - \ln(2y + 1)} \]
where \( y \) is the unique positive solution of the equation \( (y+1) \ln(2y+1) = 2(1+\ln(2))y \) [20].

References


[6] J. E. Littlewood, On polynomials $\sum^n z^m, \sum^n e^{\alpha_m} z_m, z = e^{i\beta}$, J. London Math. Soc. 41 (1966) 367–376; MR0196043 (33 #4237).


