Proof of conjectured formula for A088041

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The conjecture is that for $n \geq 4$, $2^{n-2}-1$ is the smallest integer $k > 1$ such that k^4-1 is divisible by the fourth power of an integer > 1.

Note that $k = 2^{n-2}-1$ is a fourth root of unity mod 2^n for $n \geq 4$. Indeed, mod 2^n for $n \geq 4$ there are exactly 8 fourth roots of unity, namely $1,\ 2^{n-2}-1,\ 2^{n-2}+1,\ 2^{n-1}-1,\ 2^{n-1}+1,\ 3 \cdot 2^{n-2}-1,\ 3 \cdot 2^{n-2}+1,\ 2^n-1$, and the smallest of these greater than 1 is $2^{n-2}-1$.

Thus if $a(n)$ is not $2^{n-2}-1$, it is some k with $1 < k < 2^{n-2}-1$ such that k^4-1 is divisible by p^n for some prime $p > 2$. We have $k^4-1 = (k-1)(k+1)(k^2+1)$ and the only possible common divisor of any two of these is 2, so if k^4-1 is divisible by p^n, one of $k-1$, $k+1$ and k^2+1 is divisible by p^n. If that is $k-1$ or $k+1$, we have $k+1 \geq p^n$ so $k \geq p^n-1 > 2^{n-2}-1$. If it is k^2+1, then $k \geq (p^n-1)^{1/2}$, and this is greater than $2^{n-2}-1$ if $p^n-1 > (2^{n-2}-1)^2 = 4^{n-2}-2^{n-1}+1$. That is certainly the case if $p > 4$.

The only remaining case is $p = 3$. But mod 3^n, there are only two fourth roots of unity, namely 1 and 3^n-1, and $3^n-1 > 2^{n-2}-1$. So this completes the proof of the conjecture.

Of course, $a(n) = 2^{n-2}-1$ does satisfy the recurrence $a(n) = 3a(n-1) - 2a(n-2)$ for $n \geq 6$, and it is easy to derive the generating function.