Proof of Quasi-Period 6 for Sequence A087256.

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Claim

Sequence A087256 has quasi-period 1 - 3 - 1 - x - 1 - 3 starting at n=1 except for A087256(2) = 1, where x ≥ 6 and x ≠ 7, 10.

Proof

For the respective sets of modular congruences see the link in A033496.

Let n = 6×k+j ≥ 1 with k ≥ 0 and 0 ≤ j < 6, and let $z = 2^n$.

cases j = 1, 3, 5: $2^{6\times k+j} \bmod 3 \equiv 2$ so that A087256(n) = 1.
Since 2, 8, 32 mod 3 ≡ 2, by induction
$2^{6\times(k+1)+j} \bmod 3 \equiv 2^6 \times 2^{6\times k+j} \bmod 3 \equiv 2$.

cases j = 0, 2, 4: $2^{6\times k+j} \bmod 3 \equiv 1$
Since 1, 4, 16 mod 3 ≡ 1, by induction
$2^{6\times(k+1)+j} \bmod 3 \equiv 2^6 \times 2^{6\times k+j} \bmod 3 \equiv 1$.
Now $1 < \frac{z-1}{3} < \frac{2z-2}{3} < z$ so that A087256(n) ≥ 3 when n > 2.

By induction we obtain for

j = 0: $\frac{2\times 2^6-2}{3} \equiv 0 \bmod 3$ & $\frac{2\times 2^{6\times(k+1)}-2}{3} = 2^6 \times \frac{2\times 2^{6\times k}-2}{3} + \frac{2^7-2}{3} \equiv (1\times 0 + 0) \bmod 3 \equiv 0 \bmod 3$.

j = 2: $\frac{2\times 2^2-2}{3} \equiv 0 \bmod 3$ & $\frac{2\times 2^{6\times(k+1)+2}-2}{3} = 2^6 \times \frac{2\times 2^{6\times k+2}-2}{3} + \frac{2^7-2}{3} \equiv (1\times 0 + 0) \bmod 3 \equiv 0 \bmod 3$.

j = 4: $\frac{2\times 2^4-2}{3} \equiv 1 \bmod 3$ & $\frac{2\times 2^{6\times(k+1)+4}-2}{3} = 2^6 \times \frac{2\times 2^{6\times k+4}-2}{3} + \frac{2^7-2}{3} \equiv (1\times 1 + 0) \bmod 3 \equiv 1 \bmod 3$;

$\frac{4\times 2^4-10}{9} \equiv 0 \bmod 3$ & $\frac{4\times 2^{6\times(k+1)+4}-10}{9} = 2^6 \times \frac{4\times 2^{6\times k+4}-10}{9} + \frac{(2^6-1)\times 10}{9} \equiv (1\times 0 + 0) \bmod 3 \equiv 0 \bmod 3$

furthermore, $2^4 \bmod 9 \equiv 7$ & $2^{6\times(k+1)+4} \bmod 9 \equiv 64 \times 2^{6\times k+4} \bmod 9 \equiv (1\times 7) \bmod 9 \equiv 7$;

therefore, $2\times 2^{6\times k+4} - 5 \equiv (2\times 7 - 5) \bmod 9 \equiv 0 \bmod 9$ so that $\frac{2\times 2^{6\times k+4}-5}{9}$ is an integer.

In summary, A087256(6×k) = A087256(6×k + 2) = 3 and
A087256(6×k + 4) ≥ 6 since $\frac{8z-20}{9}, \frac{4z-10}{9}, \frac{2z-5}{9}, \frac{2z-2}{3}, \frac{z-1}{3}, z$ is a trajectory to the maximum z.
Finally, the proof that A087256(6×k + 4) ≠ 7, 10 follows from the tree of fans in the link to A033496.
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