Proof of a conjecture stated in A083392

Sela Fried

Let $n \in \mathbb{N}$ and let T_n be the square matrix of size n defined by $(T_n)_{ij} = \left\lceil \frac{|i-j|}{2} \right\rceil$. Thus, T_n is a symmetric Toeplitz matrix whose first row is

$$0, 1, 1, 2, 2, 3, 3, \ldots.$$

The following theorem confirms a conjecture by Spezia stated in A083392.

Theorem 1. *Let $n \in \mathbb{N}$. Then* $\det(T_n) = (-1)^{n-1} \left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. For $n = 1, 2$ the assertion is verified by direct calculations. Thus, assume that $n \geq 3$. For each $i = n, n-1, \ldots, 3$ perform the row operation $R_i \leftarrow R_i - R_{i-2}$ and subsequently, for each $j = n, n-1, \ldots, 3$, perform the column operation $C_j \leftarrow C_j - C_{j-2}$. Let B_n denote the resulting matrix. We claim that B_n is the block matrix

$$B_n = \begin{pmatrix} A & F \\ F^T & C \end{pmatrix},$$

where A is of size 2×2, F is of size $2 \times (n-2)$, and C is of size $(n-2) \times (n-2)$, and are given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, F = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix}, C = \begin{pmatrix} -2 & -1 & 0 & \cdots & 0 \\ -1 & -2 & -1 & \ddots & \vdots \\ 0 & -1 & -2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & -2 \end{pmatrix}.$$

Indeed, let $3 \le i, j \le n$. Then

$$\begin{aligned}
(B_n)_{ij} &= (T_n)_{ij} - (T_n)_{i,j-2} - (T_n)_{i-2,j} + (T_n)_{i-2,j-2} \\
&= \left\lceil \frac{|i-j|}{2} \right\rceil - \left\lceil \frac{|i-j+2|}{2} \right\rceil - \left\lceil \frac{|i-2-j|}{2} \right\rceil + \left\lceil \frac{|i-j|}{2} \right\rceil \\
&= \begin{cases} -2, & \text{if } |i-j| = 0, \\ -1, & \text{if } |i-j| = 1, \\ 0, & \text{if } |i-j| \ge 2. \end{cases}
\end{aligned}$$

Similarly, for every $3 \le j \le n$ we have

$$\begin{aligned}
(B_n)_{1j} &= (T_n)_{1j} - (T_n)_{1,j-2} = \left\lceil \frac{j-1}{2} \right\rceil - \left\lceil \frac{j-3}{2} \right\rceil = 1, \\
(B_n)_{2j} &= (T_n)_{2j} - (T_n)_{2,j-2} = \left\lceil \frac{j-2}{2} \right\rceil - \left\lceil \frac{j-4}{2} \right\rceil = \begin{cases} 0, & \text{if } j = 3, \\ 1, & \text{if } j \ge 4. \end{cases}
\end{aligned}$$

By symmetry, the same holds for the first two columns. Finally, the 2×2 upper-left block remains unchanged during the operations and is originally given by A.

By the Schur complement formula,

$$\det(B_n) = \det(C) \det(A - FC^{-1}F^T).$$

By a well-known formula (e.g., [2]), $\det(C) = (-1)^{n-2}(n-1)$. Furthermore, it is easy to verify that the inverse of C is given by

$$(C^{-1})_{ij} = \begin{cases} (-1)^{i+j+1} \dfrac{i(n-1-j)}{n-1}, & \text{if } i \le j, \\ (-1)^{i+j+1} \dfrac{j(n-1-i)}{n-1}, & \text{if } i \ge j. \end{cases}$$

By direct calculations we obtain

$$FC^{-1}F^T = -\frac{1}{n-1} \begin{pmatrix} \left\lfloor \frac{(n-1)^2}{4} \right\rfloor & \left\lfloor \frac{(n-2)^2}{4} \right\rfloor \\ \left\lfloor \frac{(n-2)^2}{4} \right\rfloor & \left\lfloor \frac{(n-2)n}{4} \right\rfloor \end{pmatrix}.$$

Hence

$$A - FC^{-1}F^T = \begin{pmatrix} \frac{1}{n-1} \left\lfloor \frac{(n-1)^2}{4} \right\rfloor & 1 + \frac{1}{n-1} \left\lfloor \frac{(n-2)^2}{4} \right\rfloor \\ 1 + \frac{1}{n-1} \left\lfloor \frac{(n-2)^2}{4} \right\rfloor & \frac{1}{n-1} \left\lfloor \frac{(n-2)n}{4} \right\rfloor \end{pmatrix}.$$

Thus,

$$\det(A - FC^{-1}F^T) =$$
$$\frac{1}{(n-1)^2}\left\lfloor\frac{(n-1)^2}{4}\right\rfloor\left\lfloor\frac{(n-2)n}{4}\right\rfloor - \left(1+\frac{1}{n-1}\left\lfloor\frac{(n-2)^2}{4}\right\rfloor\right)\left(1+\frac{1}{n-1}\left\lfloor\frac{(n-2)^2}{4}\right\rfloor\right)$$
$$= -\frac{1}{n-1}\left\lfloor\frac{n^2}{4}\right\rfloor.$$

It follows that

$$\begin{aligned}\det(T_n) = \det(B_n) &= \det(C)\det(A - FC^{-1}F^T) = \\ &= (-1)^{n-2}(n-1)\left(-\frac{1}{n-1}\left\lfloor\frac{n^2}{4}\right\rfloor\right) \\ &= (-1)^{n-1}\left\lfloor\frac{n^2}{4}\right\rfloor. \qquad \square\end{aligned}$$

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.

[2] user17762, How to compute the determinant of a tridiagonal Toeplitz matrix?, *Math. StackExchange*, `https://math.stackexchange.com/q/267466`, 2015.