A note on the diagonals of a proper Riordan Array

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We show the exponential generating function (e.g.f.) for the n-th subdiagonal of a proper Riordan Array has the form \( \exp(cx)P_n(x) \), where \( c \) is a constant and \( P_n(x) \) is a polynomial of degree at most \( n \). The polynomials \( P_n(x) \) are the e.g.f.’s for the rows of an associated Riordan Array.

The Riordan Array associated with a pair of generating functions \( f(x) = 1 + f_1x + f_2x^2 + \cdots \) and \( g(x) = g_0 + g_1x + g_2x^2 + \cdots \) is the lower triangular array whose \( k \)-th column, \( k = 0, 1, 2, \ldots \), is generated by \( f(x)(xg(x))^k \). The matrix corresponding to the pair of generating functions \( f, g \) is denoted by \( (f(x), xg(x)) \). If we have \( g_0 \neq 0 \) then the Riordan Array is said to be proper; if \( g_0 = 0 \) then the Riordan Array is said to be stretched. In the proper-case the main diagonal is the sequence \( (1, g_0, g_0^2, g_0^3, \ldots) \), which has the e.g.f. \( \exp(g_0t) \). We wish to generalize this result to all the subdiagonals of a proper Riordan Array.

**Theorem.** With the notation as above, let \( R = (f(x), xg(x)) \) be a proper Riordan Array. Then the e.g.f. for the \( n \)-th subdiagonal of \( R \) equals \( \exp(g_0t) \times \) the e.g.f. for row \( n \) of the (possibly stretched) Riordan Array \( \tilde{R}=(f(x), g(x) - g_0) \).

**Proof.** Let \( R_{n,k} \) denote the generic element of the Riordan array \( R \). Then the \( n \)-th subdiagonal of \( R \) is the sequence \( (R_{n,0}, R_{n+1,1}, R_{n+2,2}, \ldots) \) with e.g.f.

\[
R_{n,0} + R_{n+1,1}t + R_{n+2,2}t^2 + \cdots = \sum_{k=0}^{\infty} \frac{R_{n,k} t^k}{k!} = \sum_{k=0}^{\infty} \frac{[x^n] f(x)(xg(x))^k t^k}{k!}.
\]

From the definition of a Riordan Array, \( R_{n,k} \) is the \( n \)-th coefficient of the series \( f(x)(xg(x))^k \). Using \( [x^n] \) to denote the coefficient extractor operator we see that the e.g.f. for the \( n \)-th subdiagonal of \( R \) is given by

\[
[x^n] f(x) + ([x^{n+1}] f(x)xg(x)) t + ([x^{n+2}] f(x)(xg(x))^2) \frac{t^2}{2!} + ([x^{n+3}] (xg(x))^3) \frac{t^3}{3!} + \cdots
\]

\[
= [x^n] f(x) + ([x^n] f(x)g(x)) t + ([x^n] f(x)g(x)^2) \frac{t^2}{2!} + ([x^n] g(x)^3) \frac{t^3}{3!} + \cdots
\]

\[
= [x^n] f(x) \left( 1 + g(x) t + g(x)^2 \frac{t^2}{2!} + g(x)^3 \frac{t^3}{3!} + \cdots \right)
\]

\[
= [x^n] f(x) \exp(tg(x)). \tag{1}
\]

Similarly, the e.g.f. for row \( n \) of the Riordan array \( \tilde{R} \) is equal to

\[
[x^n] f(x) + [x^n] (f(x)(g(x) - g_0)) t + [x^n] \left( f(x) (g(x) - g_0)^2 \right) \frac{t^2}{2!} + [x^n] \left( f(x) (g(x) - g_0)^3 \right) \frac{t^3}{3!} + \cdots
\]

\[
= [x^n] f(x) \exp\left( (g(x) - g_0) t \right)
\]

\[
= \exp(-g_0t) [x^n] f(x) \exp(tg(x)). \tag{2}
\]

The result now follows by comparing (1) and (2). \( \square \)
In the particular case where $R$ has the form $\left( f(x), \frac{1}{1-x} \right)$ then the array
$\tilde{R} = \left( f(x), \frac{1}{1-x} \right) = R$. So in this case the theorem says the e.g.f. for the $n$-th subdiagonal of $\tilde{R}$ equals $\exp(t)$ $\times$ the e.g.f. for row $n$ of $R$. 