The number of associative magic squares of order 7

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An associative magic square is a magic square of which sum of every pair of numbers symmetrically opposite to the center is constant. There are no associative magic squares of order 6, which has been proved by Charles Planck in 1919[1]. However, the number of associative magic squares of order 7 is not known for a long time. I have succeeded in calculating that there are 1,125,154,039,419,854,784 associative magic squares of order 7 up to rotations and reflections. The method of counting associative magic squares is described simply below.

I extend the method of Artem Ripatti's counting semi-magic squares[2] to counting associative magic squares. In his paper, he considers some equivalent classes and defines the representative patterns. Then he splits squares in two parts. Finally, he calculates each part and combines the results to get the number of semi-magic squares of order 6. I tried a similar approach, exploring possible equivalent classes, splitting squares into two parts, and combine them to get the answer.

There are some known rules transforming an associative magic square of order 7 to another. As follows:

- swapping the 1st row for the 7th row
- swapping the 2nd row for the 6th row
- swapping the 3rd row for the 5th row
- swapping the 1st row for the 2nd row and swapping 6th row for 7th row
- swapping the 2nd row for the 3rd row and swapping 5th row for 6th row

An associative magic square can be transformed to 2304(48 row rearranging and 48 column rearranging) associative magic squares by these rules.

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[1] C. Planck. Pandiagonal magics of orders 6 and 10 with minimal numbers. The Monist, 29(2):307–316, 1919.

[2] A. Ripatti. On the number of semi-magic squares of order 6. https://arxiv.org/abs/1807.02983

B	B	B	B	B	B	B
B	B	B	B	B	B	B
A	A	A	A	A	A	A
A	A	A	25	A	A	A
A	A	A	A	A	A	A
B	B	B	B	B	B	B
B	B	B	B	B	B	B

Figure 1: splitting A-part and B-part

Let canonical associative magic squares of order 7 be associative magic squares of order 7 such that there is exactly 1 canonical associative magic square in 2304 associative magic squares made from one by transforming rules. We can calculate the number of canonical associative magic squares and then multiply it by 2304/8 to get the total number of associative magic squares up to reflections and rotations.

It is an important issue how to split the square. There are some possible ways, but I selected the partition pattern as shown in Figure 1. The center value is always 25 in associative magic squares of order 7. Let C be the set of numbers in A-part and E be the set of numbers in B-part. My program calculates the number of canonical associative magic squares for each pair of C and E. Let X_{ij} be the number in the i-th row j-th cell of the square and A-profile be the tuple $(\sum_{i=3}^{5} X_{i1}, \sum_{i=3}^{5} X_{i2}, \sum_{i=3}^{5} X_{i3})$ and B-profile be the tuple $(175 - (X_{11} + X_{21} + X_{61} + X_{71}), 175 - (X_{12} + X_{22} + X_{62} + X_{72}), 175 - (X_{13} + X_{23} + X_{63} + X_{73}))$. In these definitions, a combination of an A-part arrangement and a B-part arrangement is associative magic square if and only if A-profile is equal to B-profile. Therefore, we can calculate the number of A-part canonical arrangement for each A-profile and B-part canonical arrangement for each B-profile then calculate the number of canonical associative magic squares in one pair of C and E. A-profile and B-profile can be calculated efficiently with $S = \{(s_1, s_2, ..., s_7) | s_i \in \mathbb{N}, 1 \leq s_i \leq 49, \sum_{i=1}^{7} s_i = 175\}$.

I have calculated the total number of associative magic squares of order 7 up to reflections and rotations: 1,125,154,039,419,854,784. Total time of computation is about 2 weeks with 16 core computing resources. This result fits into Walter Trump's estimated result which is published on his web page [3]. I counted the number of associated magic square of order 5 with the algorithm just like this order 7 version to get the correct answer. Thus, this result is believable enough.

[3] W. Trump. How many magic squares are there? http://www.trump.de/magic-squares/howmany.html.