On the wholeness of Sloane's A080694

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2003 March 12

1 Introduction

Sloane's sequence A080694 goes:

1, 18, 4, 18, 18, 48, 28, 54, 8, 72, 26, 36, 42, 36, ...
Let $S_n(1) = S_n(2) = 1$ and
$S_n(k+2) = |S_n(k+1) - S_n(k) - n|$.
$A080694(n)$ is the length of the period of $S_n(k)$.
$S_n(k)$ is conjectured to be periodic for any n.
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Herein I prove $S_n(k)$ is indeed periodic for any n, and A080694(n) is thus whole (defined for all n).

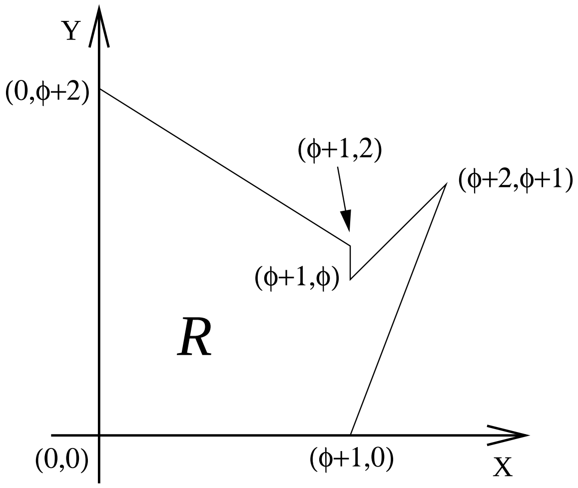

Figure 1: the region R

2 The Main Theorem

Definitions

Let $\phi = (\sqrt{5}+1)/2$; that is, the positive value for which $\phi^2 = \phi + 1$.

Let R be the region in the xy-plane bounded by the polygon with vertices $(0,0), (0,\phi+2), (\phi+1,2), (\phi+1,\phi), (\phi+2,\phi+1)$, and $(\phi,0)$. (See figure 1.)

Let F be a function that maps ordered pairs of reals to ordered pairs: $F(x,y) \mapsto (y, |y-x-1|)$.

Main Theorem: if $(x,y) \in R$, then $F(x,y) \in R$.

It is easy but tedious to verify this. To that end, partition R into the three regions U, V, and W. (See Figure 2.) In U, $y \geq x+1$, so $F(x,y) \mapsto (y, y-x-1)$; while in V and W, $y \leq x+1$, so $F(x,y) \mapsto (y, x+1-y)$.

The lower part of Figure 2 shows to where each subregion and each vertex maps. The regions U' and V' overlap in the region marked $V'U'$. ($D' = (2,\phi)$; $E' = (\phi, 2)$.)

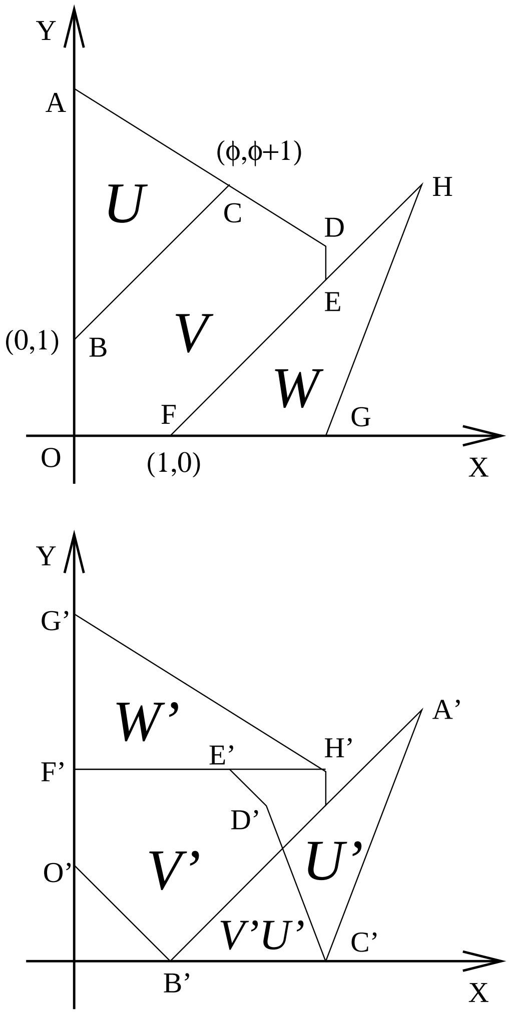

Figure 2: before and after mapping

Plainly, R could be larger or smaller. It could contain the notch at vertex E, and it's easy to calculate the intersection of AD with FH. Or R could avoid the little pentagon above region U': perhaps one can eliminate a larger, fractal-bounded region there. But we need (part of) the triangle OO'B' in the following section.

3 The series T_n and S_n

Define the series T_n:

- $T_n(1) = 1/n$,
- $T_n(2) = 1/n$
- $T_n(k+2) = |T_n(k+1) - T_n(k) - 1|$.

Note, $F(T_n(j), T_n(j+1)) \mapsto (T_n(j+1), T_n(j+2))$.

Then $(T_n(j), T_n(j+1)) \in R$ for all j. The definitions of $T_n(1)$ and $T_n(2)$ provide the induction basis; the recursion and the main theorem prove the induction step.

Therefore for all j, $0 \leq T_n(j) \leq \phi + 2$.

Define S_n:

- for all j, $S_n(j) = nT_n(j)$.

Therefore

- $S_n(1) = 1$,
- $S_n(2) = 1$
- $S_n(k+2) = |S_n(k+1) - S_n(k) - n|$.

Once notes that S_n is integer-valued, and it matches the definitions in A080694. Also:

$$0 \leq S_n(j) \leq (\phi + 2)n$$

S_n is therefore bounded. (Furthermore, the upper bound is never quite reached, since it's irrational.)

Periodicity

S_n is integral and bounded. There are only $m = \lceil(\phi+2)n\rceil$ possible values within S_n, so there can be at most m^2 distinct ordered pairs within the sequence. The sequence must repeat a pair within m^2 steps, and thereafter it is periodic. Usually, the period is much shorter than that.