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ON THE CELLULAR AUTOMATON OF ULAM AND WARBURTON

David Singmaster
87 Rodenhurst Road, London, SW4 8AF, UK

1. INTRODUCTION.

Mike Warburton's 'One-edge connections' [3] is an example of a cellular automaton or a cell growth pattern. It seems to have been first considered by Stanislaw Ulam in one of the original papers in the field [2].

We consider the square lattice as an infinite chessboard of cells, with each cell having as neighbours the four cells which share an edge with it. In generation 0, the cell at (0, 0) is transformed. In each succeeding generation, the cells which share one edge with already transformed cells are transformed. One can think of the situation as an infinite array of cells and the transformation being that they are infected, and perhaps die. Or we can think of the plane as a nutrient surface and the cells are becoming alive and propagating. The latter interpretation is more common and agrees with the idea of generation as used in Conway's Life, etc.

We first simplify by noting that the pattern has the symmetry of the square and so we need only look at one quadrant. It is convenient to take the fourth quadrant. Figure 1 shows part of the fourth quadrant with cells labelled with the generation in which they are born. Up through the sixth generation, the pattern coincides with the pattern of odd binomial coefficients, and would continue to do so if we required that life had to spread outward. After seeing Warburton's note, I did some analysis of the latter pattern and sent it to Tony Forbes since it is a well-known pattern, related to the Tower of Hanoi, the fractal known as Sierpiński's Gasket and pathological curves – see [1]. However my friend Chris Base recently asked about a pattern which had arisen in a school investigation and this was the Ulam-Warburton pattern and she pointed out that in the 7th generation, there is an inward growth, and this gets more important in higher generations. So here I do the analysis for the Ulam-Warburton pattern and determine the number of cells born in each generation.

2. THE NUMBER IN EACH GENERATION.

Let A(i) be the number of cells born in generation i. Figure 2 lists the values of A(i), up through i = 63.

Looking at Figure 1, we see that the growth from 2^n to $2^{n+1}-1$ can be viewed as three triangles, as shown in Figure 3. Triangles B and C are identical to triangle A advanced by 2^n. To see what is going on in triangle D, we need to subdivide as in Figure 4. Because the growth from point $P = (0, 2^n)$ is into virgin territory up until generation $2^{n+1}-1$, the growth is symmetric with respect to the horizontal through P. So triangle D1 is the reflection of triangle C2. However, these triangles share the common horizontal through P. So the number of cells of a given generation in the interior of D1 is one less than the number counted in C2. Also by symmetry, C2 is the reflection of C1, with a slight counting problem at the point P. Putting together all the triangles, we see that

(1) $A(2^n+i) = 2A(i) + 2[A(i)/2 - 1] = 3A(i) - 2$, for $i = 1, \ldots, 2^n-1$.

When $i = 0$, we have $A(2^n) = 2$, except $A(0) = 1$.

Since $A(i)$ is always even (except at $i = 0$), Let us set $B(i) = A(i)/2$; $B(0) = 1$. Then we have

(2) $B(2^n+i) = 3B(i) - 1$, except $B(2^n) = 1$.

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Looking at

$B(11) = 3B(3) - 1 = 3(3B(1) - 1) - 1 = 9 - 3 - 1;$

$B(12) = 3B(4) - 1 = 3 - 1;$

$B(13) = 3B(5) - 1 = 3(3B(1) - 1) - 1 = 9 - 3 - 1;$

$B(14) = 3B(6) - 1 = 3(3B(2) - 1) - 1 = 9 - 3 - 1;$

$B(15) = 3B(7) - 1 = 3(3B(3) - 1) - 1 = 3(3(3B(1) - 1) - 1) - 1 = 27 - 9 - 3 - 1;$

we see that the expressions depend just on the number, d, of ones in the binary representation of i, being $3^{d-1} - 3^{d-2} - \ldots - 1$. The tail of this is just $[3^{d-1} - 1]/2$, so we have a total of $B(i) = [3^{d-1} + 1]/2$. Doubling this gives us

(3) $A(i) = 3^{d-1} + 1$, in agreement with the values in Figure 2. This holds even when $d = 1$, i.e. $i = 2^n$, but $A(0) = 1$ is still exceptional.

I cannot yet see any simple way to describe the cells born in the i-th generation (or in all generations), nor how to determine for a given cell whether it is ever born nor in which generation. I have a rather complicated method for the latter questions, but I will postpone this until the end of this note. Such descriptions depend on the binary representation in some way.

3. THE NUMBER IN THE FIRST 2^N GENERATIONS.

Now let $C_n = \Sigma_{i=0}^{2^n-1} A(i)$ be the total number born in generations $0, 1, \ldots, 2^n-1$. A79319

We have $C_0 = 1$; $C_1 = 3$; $C_2 = 9$; $C_3 = 29$; $C_4 = 101$; $C_5 = 373$; $C_6 = 1429$; .... Either by counting as done to find equation (1), or by adding up equation (1) from $2^n + 1$ to $2^{n+1} - 1$, and using $A(0) = 1$, $A(2^n) = 2$, we find

(4) $C_{n+1} = 4C_n - (2^n - 1)$, for $n > 0$.

This is not a common type of recurrence because of the non-homogeneous terms $2^n - 1$. After some fiddling based on the idea that the solution should include terms like the non-homogeneous part, I realised I could eliminate this part by considering $C_n = D_n + a2^n + b$.

This yields $D_{n+1} + a2^{n+1} + b = 4D_n + 4a2^n + 4b - 2^n + 1$. Setting $b = -1/3$ makes the constant part cancel out, and setting $a = 1$ makes the 2^n part cancel out. We are then left with $D_{n+1} = 4D_n$, whose solution is obviously $D_n = \alpha 4^n$. Hence $C_n = \alpha 4^n + 2^n - 1/3$. Using $C_1 = 3$, we find $\alpha = 1/3$, so

(5) $\quad C_n = (4^n-1)/3 + 2^n$.

The total number of cells in levels 0 through $2^n - 1$ is $1 + 2 + ... + 2^n = 2^n(2^n+1)/2 = t(2^n)$, the 2^n-th triangular number. Hence the density of live cells up through $2^n - 1$ is $C_n/t(2^n) = 2/3 - 1/3 \cdot 2^{n-1} + 1/(2^n+1)$, which is asymptotic to 2/3.

To relate to Mike Warburton's expression, we let E_n be the number in his pattern for levels 0 through $2^n - 1$. We have $E_0 = 1$; $E_1 = 5$; $E_2 = 21$; $E_3 = 85$; $E_4 = 341$; .... Note that Warburton includes the level 2^n, which adds four to E_n. Since the whole figure is four quadrants, with some overlap along the axes, we find $E_n = 4C_n - 3 - 4(2^n-1)$, and using equation (5), we have

(6) $\quad E_n = (4^{n+1} - 1)/3$.

Adding four and slightly rearranging gives $E_n + 4 = 4 \cdot 4^n/3 + 11/3$, which is Warburton's expression.

Surprisingly, neither A(i) nor B(i) appear in Sloane's On-Line Encyclopedia of Integer Sequences (www.research.att.com/cgi-bin/access.cgi/as/njas/sequences/eismum.cgi) and E_n appears as in equation (6) but only in quite different contexts.

4. CONCLUDING REMARKS.

Let me now describe my somewhat complicated process of determining whether and when a given cell becomes alive. Basically we use the recursive observations about Figures 3 and 4 to reduce the coordinates. Let a cell have coordinates (i, j) where j is taken positive in the downward direction. Let G(i, j) be the generation number in which cell (i, j) is born. Since Figure 1 is symmetric with respect to its diagonal, i.e. G(i, j) = G(j, i), we can assume $i \leq j$ and we need only look at the lower part of Figure 1.

The growth from generation 2^n to generation $2^{n+1} - 1$ all lies in the triangles B, C, D of Figure 3, i.e. where $2^n \leq i + j \leq 2^{n+1} - 1$.

If $j \geq 2^n$, we are in triangle C and we have

(7) $\quad G(i, j) = 2^n + G(i, j-2^n)$.

But if $i \leq j < 2^n$, then we are interior to triangle D1 and reflection in the horizontal through P gives $G(i, j) = G(i, 2^{n+1}-j)$. Use of (7) then gives us

(8) $\quad G(i, j) = 2^n + G(i, 2^{n+1}-j)$.

Rotating triangle D1 into triangle C1 gives us $G(i, j) = G(2^n-j, 2^n+i)$ and use of (7) gives us

(9) $\quad G(i, j) = 2^n + G(2^n-j, i)$,

which is a symmetric form of (8). However, when $i = j = 2^{n-1}$, both transformations bring us back to the same point we started with. If $i = j = 0$, we are at the end of our process, but if $i = j > 0$, our point is never born. Other general rules such as $G(0, j) = j$ and

$G(i, 2^n\text{-}1\text{-}i) = 2^n - 1$ help shorten any calculation. One can describe these rules in terms of the binary expansions of i and j, but the dichotomy of the rules and the use of 2s complement in (8) or (9) makes it quite unclear what the overall result of the process will be.

As a final remark, observe that the pattern of unborn cells is the same fractal as the pattern of born cells, but rotated by 45° and shrunk by a factor of $\sqrt{2}$.

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London, 27 Jan - 1 Feb 2003, my #497.

G(i, j)

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00 01 02 03 04 05 06 07 08 09 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32
01 -- 03 -- 05 -- 07 -- 09 -- 11 -- 13 -- 15 -- 17 -- 19 -- 21 -- 23 -- 25 -- 27 -- 29 -- 31 -- 33
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07 -- 11 -- 13 -- 15 -- 15 -- 31 -- 29 -- 27 -- 23 -- 27 -- 29 -- 31 -- 31 -- 47 -- 45 -- 43 -- 39
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62 63 --
63 --
64
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This is A79314

$$= \frac{A147582(n)}{4} + 1$$

Not the Sierpinski gasket because of the 06's

FIGURE 1.

FIGURE 2.

i	A(i)	i	A(i)	i	A(i)	i	A(i)
0	1	16	2	32	2	48	4
1	2	17	4	33	4	49	10
2	2	18	4	34	4	50	10
3	4	19	10	35	10	51	28
4	2	20	4	36	4	52	10
5	4	21	10	37	10	53	28
6	4	22	10	38	10	54	28
7	10	23	28	39	28	55	82
8	2	24	4	40	4	56	10
9	4	25	10	41	10	57	28
10	4	26	10	42	10	58	28
11	10	27	28	43	28	59	82
12	4	28	10	44	10	60	28
13	10	29	28	45	28	61	82
14	10	30	28	46	28	62	82
15	28	31	82	47	82	63	242

A79314

$$= \frac{A147582(n)}{4} + 1$$

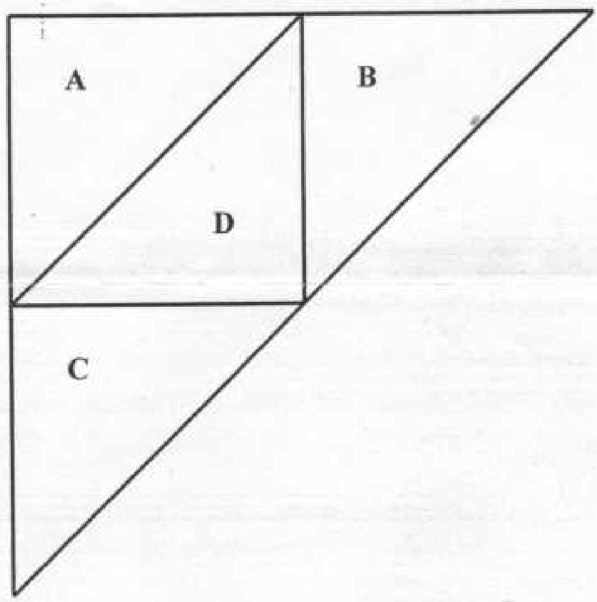

FIGURE 3.

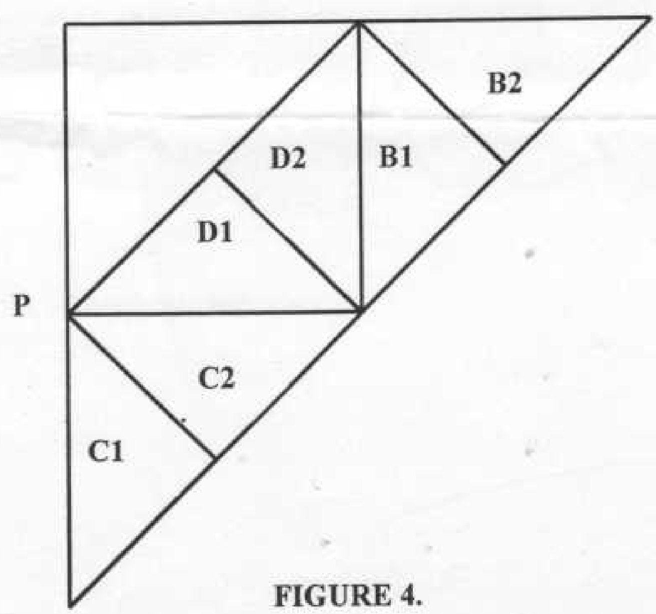

FIGURE 4.