On Sun's conjectures for determinants of $[(i-j)^m+\delta_{ij}]_{1\le i,j\le n}$

Sela Fried

Abstract

Let $A_n^{(m)} = [(i-j)^m+\delta_{ij}]_{1\le i,j\le n}$. We evaluate $\det(A_n^{(m)})$ in terms of power sums and prove related conjectures recorded by Sun in A355175 and A355326 in the On-Line Encyclopedia of Integer Sequences.

1 Introduction

In [6] Wang and Sun evaluated several Toeplitz-type determinants, one of which is the determinant of the matrix $[i-j+\delta_{ij}]_{1\le i,j\le n}$, where δ_{ij} is the Kronecker delta. In A355175 [5] Sun conjectured a closed-form formula for the determinant of the matrix $[(i-j)^2+\delta_{ij}]_{1\le i,j\le n}$. In A355326 Sun conjectured a closed-form formula for the determinant of the matrix $[(i-j)^3+\delta_{ij}]_{1\le i,j\le n}$ and made the general conjecture that for every natural number m, the determinant of the matrix $[(i-j)^m+\delta_{ij}]_{1\le i,j\le n}$ has the form $1+n^2(n^2-1)P(n)$, where $P(n)$ is a rational polynomial in n with degree $(m+1)^2-4$.

Following an approach suggested by a referee in [6], namely, by using Sylvester's determinant theorem, we resolve all of Sun's conjectures mentioned above.

2 Main results

Let $\mathbb{N}=\{1,2,\ldots\}$ denote the set of natural numbers and set $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $n\in\mathbb{N}_0$ let $B_n(x)$ denote the nth Bernoulli polynomial and for a polynomial $p(x)$ let $[x^n]p(x)$ denote the coefficient of x^n in $p(x)$. Finally, for $r\in\mathbb{N}_0$ and $n\in\mathbb{N}$, set $s_r(n)=\sum_{k=1}^n k^r$.

Theorem 1. *For* $m, n \in \mathbb{N}$ *let* $A_n^{(m)} = [(i-j)^m + \delta_{ij}]_{1 \le i,j \le n}$. *Then*

$$\det(A_n^{(m)}) = \det\left(\left[\delta_{ab} + (-1)^a \binom{m}{a} s_{a+m-b}(n)\right]_{0 \le a,b \le m}\right). \tag{1}$$

Proof. We have $A_n^{(m)} = I_n + M_n^{(m)}$, where $M_n^{(m)}$ is the square matrix of size n defined by $(M_n^{(m)})_{ij} = (i-j)^m$. For $r \in \mathbb{N}_0$ set $\mathbf{v}_r = (1^r, 2^r, \ldots, n^r)^T$. By the binomial theorem,

$$(i-j)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} i^{m-k} j^k.$$

Thus,

$$M_n^{(m)} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathbf{v}_{m-k} \mathbf{v}_k^T.$$

Let U and V be the two matrices of size $n \times (m+1)$ given column-wise by $U_k = \mathbf{v}_{m-k}$ and $V_k = (-1)^k \binom{m}{k} \mathbf{v}_k$, $k = 0, 1, \ldots, m$, respectively. One easily verifies that $M_n^{(m)} = UV^T$ and therefore $A_n^{(m)} = I_n + UV^T$. By Sylvester's determinant theorem (e.g., [4, (8.1.16)]),

$$\det(A_n^{(m)}) = \det(I_{m+1} + V^T U).$$

Now, for every $0 \le a, b \le m$ we have

$$(V^T U)_{ab} = V_a^T U_b = \left((-1)^a \binom{m}{a} \mathbf{v}_a\right)^T \mathbf{v}_{m-b} = (-1)^a \binom{m}{a} s_{a+m-b}(n),$$

and the assertion follows. □

Theorem 2. *Consider the matrix*

$$R_m(n) = \left[\delta_{ab} + (-1)^a \binom{m}{a} s_{a+m-b}(n)\right]_{0 \le a,b \le m}, \tag{2}$$

and let $D_m(n) = \det(R_m(n))$. *Then there exists a polynomial* $P_m(n) \in \mathbb{Q}[n]$ *of degree* $(m+1)^2 - 4$ *such that*

$$D_m(n) = 1 + n^2(n^2-1) P_m(n).$$

Proof. Let $r \in \mathbb{N}_0$. By Faulhaber's formulas, $s_r(n)$ is a rational polynomial in the variable n. Now, by (1), $D_m(1) = \det(A_1^{(m)}) = 1$. Thus, $D_m(n) - 1$ is divisible by $n - 1$. It is well-known (e.g., [1, (7.79)]) that

$$s_r(n) = \frac{B_{r+1}(n+1) - B_{r+1}(0)}{r+1} - 0^r. \tag{3}$$

Thus, $s_r(-1) = 0$ for every $r \geq 1$ and therefore $s_{a+m-b}(-1) = 0$ for every $0 \leq a, b \leq m$, except when $a = 0$ and $b = m$, and, in this case, $s_0(-1) = -1$. Consequently, $R_m(-1) = I_{m+1} + E$, where E has a single nonzero entry $(E)_{0m} = -1$. Hence, $D_m(-1) = \det(R_m(-1)) = 1$ and therefore $D_m(n) - 1$ is divisible by $n + 1$. This concludes the proof that $n^2 - 1$ divides $D_m(n) - 1$.

We now wish to prove that $D_m(n) - 1$ is divisible by n^2. First, notice that since $s_r(0) = 0$, we have $R_m(0) = I_{m+1}$ and therefore $D_m(0) = \det(R_m(0)) = 1$. This shows that n divides $D_m(n) - 1$. Now, By Jacobi's formula (e.g., [3, (8) on p. 170]),

$$(\det(R_m(n)))' = \operatorname{tr}(\operatorname{adj}(R_m(n))R'_m(n)).$$

Thus,

$$\begin{aligned}
(\det(R_m(0)))' &= \operatorname{tr}(\operatorname{adj}(I_{m+1})R'_m(0)) \\
&= \operatorname{tr}(R'_m(0)) \\
&= \sum_{a=0}^{m} (R'_m(0))_{aa} \\
&= s'_m(0) \sum_{a=0}^{m} (-1)^a \binom{m}{a} \\
&= s'_m(0)(1-1)^m \\
&= 0.
\end{aligned}$$

This concludes the proof that n^2 divides $D_m(n) - 1$.

We shall now prove that $\deg(P_m(n)) = (m+1)^2 - 4$. By (3), $\deg(s_r(n)) = r + 1$. Thus, the degree matrix corresponding to $R_m(n)$ is given by

$$[a + m - b + 1]_{0 \leq a, b \leq m}.$$

Let π be a permutation of $\{0, 1, \ldots, m\}$. We have

$$\sum_{a=0}^{m} (a + m - \pi(a) + 1) = \sum_{a=0}^{m} (a + m + 1) - \sum_{a=0}^{m} \pi(a) = (m+1)^2,$$

It follows that $\deg(D_m(n)) \leq (m+1)^2$. On the other hand, we have $[n^{a+m-b+1}](R_m(n))_{ab} = (-1)^a\binom{m}{a}\frac{1}{a+m-b+1}$. Thus, $[n^{(m+1)^2}]\det(R_m(n))$ is equal to the determinant of the matrix

$$\left[(-1)^a\binom{m}{a}\frac{1}{a+m-b+1}\right]_{0\leq a,b\leq m},$$

which is, up to row factors, the Cauchy matrix (e.g., [2, 0.9.12])

$$\left[\frac{1}{x_a+y_b}\right]_{0\leq a,b\leq m},$$

with $x_a = a$ and $y_b = m-b+1$. Since the x_as and the y_bs are distinct, the determinant of the Cauchy matrix is nonzero. It follows that $\deg(D_m(n)) = (m+1)^2$. Thus, factoring out the term $n^2(n^2-1)$ of degree 4 from $D_m(n)-1$, the remaining polynomial $P_m(n)$ has degree $(m+1)^2-4$. $\square$

Corollary 1 (A079034)**.** *Let* $n \in \mathbb{N}$*. Then*

$$\det(A_n^{(1)}) = 1 + \frac{n^2(n^2-1)}{12}.$$

Proof. By (1),

$$\det(A_n^{(1)}) = \det\begin{pmatrix} 1+s_1(n) & s_0(n) \\ -s_2(n) & 1-s_1(n)\end{pmatrix}. \qquad \square$$

Corollary 2 (A355175)**.** *Let* $n \in \mathbb{N}$*. Then*

$$\det(A_n^{(2)}) = 1 + n^2(n^2-1)\frac{n^5-5n^3-36n^2+4n+54}{1080}.$$

Proof. By (1),

$$\det(A_n^{(2)}) = \det\begin{pmatrix} 1+s_2(n) & s_0(n) & -2s_1(n) \\ s_4(n) & 1+s_2(n) & -2s_3(n) \\ s_3(n) & s_1(n) & 1-2s_2(n)\end{pmatrix}. \qquad \square$$

Corollary 3 (A355326)**.** *Let* $n \in \mathbb{N}$*. Then*

$$\det(A_n^{(3)}) = 1+n^2(n^2-1)\frac{n^{12}-19n^{10}+123n^8-337n^6+12376n^4-44144n^2+40000}{672000}.$$

Proof. By (1),

$$\det(A_n^{(3)}) = \det\begin{pmatrix} 1+s_3(n) & s_2(n) & s_1(n) & s_0(n) \\ -3s_4(n) & 1-3s_3(n) & -3s_2(n) & -3s_1(n) \\ 3s_5(n) & 3s_4(n) & 1+3s_3(n) & 3s_2(n) \\ -s_6(n) & -s_5(n) & -s_4(n) & 1-s_3(n)\end{pmatrix}. \qquad \square$$

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed., Addison–Wesley, Reading, MA, 1994.

[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge Univ. Press, Cambridge, 2013.

[3] J. R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, rev. ed., Wiley, Chichester, 1999.

[4] C. Pozrikidis, *An Introduction to Grids, Graphs, and Networks*, Oxford Univ. Press, Oxford, 2014.

[5] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.

[6] H. Wang and Z.-W. Sun, Evaluations of some Toeplitz-type determinants, arxiv preprint, *arXiv*:2206.12317 [math.NT], 2023. Available at 10.48550/arXiv.2206.12317.