A condition on the odd divisors of n for the symmetric representation of σ(n) to have an odd number of parts.

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Notations and Terminology

$n = 2^m \times q$, $m \geq 0$, q odd.

$r_n = \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right)\right\rfloor$

$D_n = \{ s : s|q \text{ and } s \times 2^{m+1} > r_n \}$

$E_n = \{ s : s|q \text{ and } s \times 2^{m+1} \leq r_n \}$

$d_n = \min(D_n)$ and $e_n = \max(E_n)$ when the respective sets are not empty.

We use SRS for the phrase "symmetric representation of sigma"; SRS(σ(n)) denotes its area (which equals σ(n), see A280851), and #SRS(σ(n)) denotes the "number of parts of the symmetric representation of σ(n)".

Theorem

$D_n \neq \emptyset$ is equivalent to #SRS(σ(n)) is odd.

Proof

Suppose $D_n \neq \emptyset$. If $E_n = \emptyset$ then $1 \in D_n$ and 1's in row n of A237048 occur only at odd indices so that all entries in row n of A249223 are positive, i.e. #SRS(σ(n)) = 1. Let $E_n \neq \emptyset$ and suppose $e_n \times 2^{m+1} < d_n$. Then the number of 1's at odd indices equals the number of 1's at even indices for indices less than d_n so that there is a 0 at index d_n - 1 in row n of A237048 as well as A249223. Therefore, all indices from d_n through r_n in row n of A249223 are positive defining a part of SRS that crosses the diagonal which implies that #SRS(σ(n)) is odd. Finally, if $E_n \neq \emptyset$ and $d_n < e_n \times 2^{m+1} \leq r_n$ suppose that $s_1 < s_2 < \ldots < s_k = e_n$, k ≥ 1, are the elements in E_n satisfying

$$s_1 < s_2 < \ldots < s_k = e_n < d_n < s_1 \times 2^{m+1} < \ldots < s_k \times 2^{m+1} = e_n \times 2^{m+1} \leq r_n.$$

Since odd divisors $s_1, s_2, \ldots, s_k$ are matched only by even indices larger than d_n, the value at index d_n in row n of A249223 must equal k+1. Because at only k even indices the value in row n of A237048 is 1 between indices d_n and r_n , all indices from d_n through r_n in row n of A249223 are positive defining a part of SRS that crosses the diagonal which implies that #SRS(σ(n)) is odd.

Conversely, if #SRS(σ(n)) is odd then there is an odd divisor s of q, $1 \leq s \leq r_n$, such that for all indices i, $s \leq i \leq r_n$, the values in row n of A249223 are positive with the value at s being 1 and 0 at s-1 if s>1. If $s \times 2^{m+1} \leq r_n$ were true then A249223 would have value 0 at index $s \times 2^{m+1}$ since all 1's at all odd indices through s would be matched, contradicting the choice of s. Therefore, $s \times 2^{m+1} > r_n$ and $D_n \neq \emptyset$.