On the Symmetric Spectrum of Odd Divisors of a Number

Hartmut F. W. Höft
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Abstract

The starting point for this note is the irregular triangle A237048 of 0's and 1's. We prove that the 1's in the n-th row of the triangle represent the odd divisors of n (A001227) and establish a formula for the positions of the 1's in the n-th row. The rows of the related irregular triangle A237593 define a sequence of non-crossing symmetric Dyck paths in the first quadrant. We refer to the regions between the (n-1)-st and n-th Dyck path as the symmetric spectrum of odd divisors of n. It is conjectured that the sum of the areas in this spectrum equals σ(n), the sum of all divisors of n (A000203).

This note contains two main results. In Theorem 3 we characterize those n with an odd or even number of regions in their spectrum, respectively, and show that two pairs of sequences are identical: A071561 = A241561 and A071562 = A241562. In Theorem 6 we prove the conjecture about σ(n) when n has an even number of regions, all of width 1 (A241008); the companion for odd number of regions of width 1 (A241010) has already been established.

Notations, Terminology & Basic

In this section we introduce notations for a number of sequences in OEIS, establish formulas for them and prove some basic properties about them. Some of the definitions are verbatim copies from OEIS except for the symbolic names for the sequences.

DEFINITION 1: Triangle D (A237048)

Irregular triangle read by rows: $d_{n,k}$, for all n ≥ 1 and k ≥ 1, in which column k lists 1's interleaved with k-1 0's, and the first element of column k is in row $\frac{k\times(k+1)}{2}$.

The length of the n-th row is $r_n = \left\lfloor \frac{1}{2}\left(\sqrt{8n+1} - 1\right)\right\rfloor$ (A003056).

For convenience in computations later on we define $d_{n,k} = 0$ for all n, k ∈ ℕ with 1 ≤ n and $r_n < k$.

DEFINITION 2:

$$\Delta(n,k) = \begin{cases} 1 & \text{if } k \mid n \\ 0 & \text{otherwise} \end{cases}, \text{ for all n, k} \in \mathbb{N} \text{ with } 1 \le k \le n,$$

denotes the **characteristic function of the divisor function**.

Simple algebraic manipulations establish the following inequalities and identities.

LEMMA 1:

(a) For all n ∈ ℕ with $\frac{k\times(k+1)}{2} \le n < \frac{(k+1)\times(k+2)}{2}$, $r_n = k$.

(b) For all n ∈ ℕ, n ≥ 1, $\sqrt{n/2} \le r_n < \sqrt{2n}$.

(c) For all n, k ∈ ℕ with 1 ≤ n and 1 ≤ k ≤ r_n, $d_{n,k} = \begin{cases} \Delta(n,k) & \text{if } k \text{ is odd} \\ \Delta\left(n - \frac{k}{2}, k\right) & \text{if } k \text{ is even} \end{cases}$.

(d) Let n ∈ ℕ be of the form $n = 2^m \times s \times t$, with s, t ∈ ℕ odd and m ∈ ℕ, m ≥ 0.
 (i) $2^{m+1} \times s \le r_n \iff r_n < t$.
 (ii) $2^{m+1} \times s < t \Rightarrow r_n < t$.

(e) Let n ∈ ℕ be of the form $n = 2^m \times q$ where m, q ∈ ℕ, m ≥ 0 and q is odd.
 The $d_{n,k} = 1$ precisely when

(i) $1 \le k \le r_n$ is an odd divisor of n, or

(ii) $k = 2^{m+1} \times s$ when $n = 2^m \times s \times t$, with s, t ∈ ℕ odd, $1 \le s < 2^{m+1} \times s \le r_n < t$.

PROPOSITION 1:

The sum of entries in the n-th row of triangle D (A237048) equals the number of odd divisors of n (A001227).

PROOF:

Lemmas 1(d) & 1(e) say that each odd divisor t of $n = 2^m \times s \times t$ greater than r_n is uniquely paired with the even position $2^{m+1} \times s$ less than or equal to r_n. Therefore, $d_{n,k} = 1$ for the index pair $n = 2^m \times s \times t$ and $k = 2^{m+1} \times s$. Similarly, $d_{n,k} = 1$ for the index pair $n = 2^m \times s \times t$ and s, for any odd divisor $s \le r_n$. The claim follows.

■

DEFINITION 3: Triangle W (A249223 & A250068)

Irregular triangle of partial alternating sums of row entries of triangle A237048, read by rows:
$w_{n,k} = \sum_{j=1}^{k} (-1)^{j+1} \times d_{n,j}$, for all n, k ∈ ℕ, n ≥ 1 and $1 \le k \le r_n$.

We call number $w_n = \max(w_{n,k} \mid 1 \le k \le r_n)$, for all n ∈ ℕ, n ≥ 1, the **symmetric width for n** or simply **width**.

Since an empty sum has value zero, it is convenient to define $w_{n,0} = 0$ for all n ≥ 1. Furthermore, as we did with $d_{n,k}$, we define $w_{n,k} = 0$ for all n, k ∈ ℕ with 1 ≤ n and $r_n < k$.

PROPOSITION 2:

(a) For all n, k ∈ ℕ, n ≥ 1 and $1 \le k \le r_n$: $w_{n,k} \ge 0$.

(b) For all n, k ∈ ℕ, n ≥ 1 and $1 \le k \le r_n$: $w_{n+1,k+1} - w_{n+1,k-1} = (-1)^{k+1} \times (d_{n+1,k} - d_{n+1,k+1})$.

(c) For all n ∈ ℕ, n ≥ 1: $w_n \ge 1$.

PROOF of (a):

$w_{n,1} = 1$ since $d_{n,1} = 1$. Let $j = 2^{m+1} \times s$ with s odd and $d_{n,j} = 1$. Then also $d_{n,s} = 1$. Therefore, the number of 1's in odd-numbered columns prior to column j is at least as large as the number of ones in even-numbered columns through j.

■

The rows of the two irregular triangles D and W satisfy the following partial periodicity.

PROPOSITION 3:

For all n ∈ ℕ, n ≥ 1, let $m = \mathrm{lcm}(i \mid 1 \le i \le r_n)$. Then for all $1 \le k \le r_n$:
$d_{n+m,k} = d_{n,k}$ and $w_{n+m,k} = w_{n,k}$.

PROOF:

Observe that k | (n+m) precisely when k | n and k | $(n - \frac{k}{2} + m)$ precisely when k | $(n - \frac{k}{2})$, proving the first identity. The second follows from $w_{n+m,k} = \sum_{j=1}^{k} (-1)^{j+1} \times d_{n+m,j} = \sum_{j=1}^{k} (-1)^{j+1} \times d_{n,j} = w_{n,k}$.

■

DEFINITION 4: Triangle E (A235791):

Irregular triangle read by rows: $e_{n,k}$, n ≥ 1, k ≥ 1, in which column k lists k copies of every positive integer in nondecreasing order, and the first element of column k is in row $\frac{k \times (k+1)}{2}$.

DEFINITION 5: Triangle F (A237591):

Irregular triangle read by rows : $f_{n,k} = e_{n,k} - e_{n,k+1}$, for all n ≥ 1 and 1 ≤ k ≤ r_n.
By Lemma 2 (c) the entries $f_{n,k}$ are well defined in the stated index ranges.

LEMMA 2:

For all 1 ≤ n and 1 ≤ k ≤ r_n , or equivalently, for $n \geq \frac{k \times (k+1)}{2}$:

(a) $e_{n,k} = \sum_{i=1}^{n} d_{i,k}$.
(b) $e_{n,k} = \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil > 0$.
(c) $e_{n,r_n+1} = 0$.
(d) $f_{n,k} > 0$.
(e) $\sum_{k=1}^{r_n} f_{n,k} = n$.
(f) $\sum(f_{n,k} \mid 1 \leq k \leq r_n \;\&\; k \text{ even}) = n - \sum(f_{n,k} \mid 1 \leq k \leq r_n \;\&\; k \text{ odd})$.
(g) $f_{1,1} = 1$ and $f_{n+1,k} = f_{n,k} + d_{n+1,k} - d_{n+1,k+1}$
(h) $f_{1,1} = 1$ and $f_{n+1,k} = f_{n,k} + (-1)^{k+1} \times (w_{n+1,k+1} - w_{n+1,k-1})$.

PROOF:

We prove parts (b) and (d), the others are simple computations. For part (b) observe that the elements in column k of the triangle E start in position $\frac{k \times (k+1)}{2}$ and are repeated k times. There are $n - \frac{k \times (k+1)}{2} + 1$ non-zero entries in column k from row $\frac{k \times (k+1)}{2}$ through row n inclusive. Therefore the number in the k-th column of row n of the triangle is $\frac{1}{k}$-th of that count of non-zero entries, rounded up:

$e_{n,k} = \left\lceil \frac{1}{k} \left(n + 1 - \frac{k}{2} (k+1) \right) \right\rceil = \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil$ for 1 ≤ n and 1 ≤ k ≤ r_n.

For part (d) observe that $f_{n,k} = \left\lceil \frac{n+1}{k} - \frac{k+1}{2} \right\rceil - \left\lceil \frac{n+1}{k+1} - \frac{k+2}{2} \right\rceil$
$= \left\lceil \frac{2kn + 2n + 2k + 2 - k^3 - 2k^2 - k}{2k(k+1)} \right\rceil - \left\lceil \frac{2kn + 2k - k^3 - 3k^2 - 2k}{2k(k+1)} \right\rceil$

Let $a = \frac{2kn + 2k - k^3 - 2k^2 - k}{2k(k+1)}$ and let $a = q + \delta$ with $q \in \mathbb{N}$ and $0 \leq \delta < 1$.

Since $n \geq \frac{k \times (k+1)}{2}$, we have $\frac{n+1}{k(k+1)} \geq \frac{k(k+1)+2}{2k(k+1)} \geq \frac{1}{2} + \frac{1}{k(k+1)}$.

Case 1: $0 \leq \delta \leq \frac{1}{2}$

$f_{n,k} = \left\lceil q + \delta + \frac{n+1}{k(k+1)} \right\rceil - \left\lceil q + \delta - \frac{1}{2} \right\rceil = \left\lceil \delta + \frac{n+1}{k(k+1)} \right\rceil - \left\lceil \delta - \frac{1}{2} \right\rceil \geq \left\lceil \delta + \frac{1}{2} + \frac{1}{k(k+1)} \right\rceil \geq 1$

Case 2: $\frac{1}{2} < \delta < 1$

In this case, $\delta + \frac{1}{2} + \frac{1}{k(k+1)} > 1$ so that $\left\lceil \delta + \frac{1}{2} + \frac{1}{k(k+1)} \right\rceil \geq 2$, and $\left\lceil \delta - \frac{1}{2} \right\rceil = 1$. Therefore,

$f_{n,k} = \left\lceil \delta + \frac{n+1}{k(k+1)} \right\rceil - \left\lceil \delta - \frac{1}{2} \right\rceil \geq 2 - 1 = 1$.

■

LEMMA 3:

(a) For all n ≥ 1 and all 1 ≤ k ≤ $r_n = r_{n+1}$:
$\sum(f_{n+1,i} \mid 1 \leq i \leq k \leq r_{n+1} \;\&\; i \text{ odd}) \geq \sum(f_{n,i} \mid 1 \leq i \leq k \leq r_n \;\&\; i \text{ odd})$ and
$\sum(f_{n+1,i} \mid 1 \leq i \leq k \leq r_{n+1} \;\&\; i \text{ even}) \leq \sum(f_{n,i} \mid 1 \leq i \leq k \leq r_n \;\&\; i \text{ even})$

(b) For all n ≥ 1 and $r_{n+1} = r_n + 1$:
$\sum(f_{n+1,k} \mid 1 \leq k \leq r_{n+1} \;\&\; k \text{ odd}) \geq \sum(f_{n,k} \mid 1 \leq k \leq r_n \;\&\; k \text{ odd})$ and
$\sum(f_{n+1,k} \mid 1 \leq k \leq r_{n+1} \;\&\; k \text{ even}) \leq \sum(f_{n,k} \mid 1 \leq k \leq r_n \;\&\; k \text{ even}) + 1$

PROOF:

As a representative we show the argument for the last of the four inequalities. When $r_{n+1} = r_n + 1$ is even then we get with Lemma 2(f) and the inequality for odd indices above:

$\sum(f_{n+1,k} \mid 1 \le k \le r_{n+1}$ & k even) = n + 1 - $\sum(f_{n+1,k} \mid 1 \le k \le r_{n+1}$ & k odd)
= ($\sum(f_{n,k} \mid 1 \le k \le r_n$ & k even) + $\sum(f_{n,k} \mid 1 \le k \le r_n$ & k odd)) + 1 - $\sum(f_{n+1,k} \mid 1 \le k \le r_{n+1}$ & k odd)
= $\sum(f_{n,k} \mid 1 \le k \le r_n$ & k even) + 1 + ($\sum(f_{n,k} \mid 1 \le k \le r_n$ & k odd) - $\sum(f_{n+1,k} \mid 1 \le k \le r_{n+1}$ & k odd))
≤ $\sum(f_{n,k} \mid 1 \le k \le r_n$ & k even) + 1
■

Symmetric Dyck Paths and the Symmetric Spectrum of Odd Divisors for a Number n

The irregular triangles D, W, E and F define only half of the numbers needed to describe the geometric figures that encode the divisor function σ. Therefore we symmetrically extend W and F which encode widths and sidelengths of rectangular areas.

DEFINITION 6: Triangle $\overline{W}$ (A262045 = A249223 ∪ Reverse(A249223))

Irregular triangle read by rows : $\overline{w}_{n,k} = \begin{cases} w_{n,k} & \text{for } 1 \le k \le r_n \\ w_{n,2r_n+1-k} & \text{for } r_n < k \le 2r_n \end{cases}$, for all n ≥ 1.

DEFINITION 7: Triangle $\overline{F}$ (A237593 = A237591 ∪ Reverse(A237591))

Irregular triangle read by rows : $\overline{f}_{n,k} = \begin{cases} f_{n,k} & \text{for } 1 \le k \le r_n \\ f_{n,2r_n+1-k} & \text{for } r_n < k \le 2r_n \end{cases}$, for all n ≥ 1.

Different definitions for Dyck paths appear in the literature. We use the convention shown in Figures 1, 2 & 3 of A237593 as well as in other sequences in OEIS rather than the first illustration of a Dyck path shown in the entry of that sequence.

DEFINITION 8:

A **Dyck path** is the staircase path (horizontal/right - vertical/down - horizontal/right - vertical/down …) in the first quadrant with legs of integral lengths that starts at the y-axis and ends on the x-axis. A Dyck path that is symmetric about the diagonal is called a **symmetric Dyck path**.

A symmetric Dyck path starts at point (0, n) and ends at point (n, 0) and has length 2×n, for some n ∈ ℕ. By construction the Dyck path whose legs are defined by the numbers in the n-th row of triangle $\overline{F}$ is symmetric.

THEOREM 1:

For all n ∈ ℕ, n ≥ 1, the symmetric Dyck path defined by the n-th row of triangle $\overline{F}$ always is outside the region of the symmetric Dyck path defined by the (n-1)-st row of triangle $\overline{F}$. Two adjacent Dyck paths may touch or share parts of their edges, but they never cross each other.

PROOF:

By Lemma 2 (e) the sum of the lengths of the first half of the n-th row of triangle $\overline{F}$ is n. We start the Dyck path defined by the n-th row of triangle $\overline{F}$ at coordinate (0, n). Numbers $\overline{f}_{n,k}$ for odd values of k are interpreted as the lengths of the k-th horizontal leg and numbers $\overline{f}_{n,k}$ for even values of k are declared as the lengths of the k-th vertical leg of the path. Since each row in triangle $\overline{F}$ is symmetric about its middle it defines a symmetric Dyck path of length 2×n from (0, n) to (n, 0). Note also that by Lemma 2 (e) & (f), number z = ($f_{n,k} \mid 1 \le k \le r_n$ & k even) = n - $\sum(f_{n,k} \mid 1 \le k \le r_n$ & k odd) defines the midpoint of the path at coordinate (z, z).

Lemma 3 establishes that the corresponding partial sums of the horizontal legs of the n-th path extend at least as far right as those of the (n-1)-st path and that the corresponding partial sums of the vertical legs of the n-th path extend at most as far down as those of the (n-1)-st path, proving the claim. Since the inequalities in Lemma 3 are non-strict, two adjacents paths may meet, and the second inequality of

Lemma 3 (b) implies that the two adjacent paths may meet in a point on the diagonal.
■

DEFINITION 9: Triangle S (A262048)

Irregular triangle read by rows: $s_{n,k} = \overline{w}_{n,k} \times \overline{f}_{n,k}$, for all n ≥ 1 and 1 ≤ k ≤ $2 \times r_n$.
We call the symmetric sequence of numbers $s_n = (s_{n,1}, \ldots, s_{n,2 \times r_n})$ of the n-th row the **symmetric spectrum of odd divisors for n**, and we call each of the maximal disjoint blocks of non-zero members of s_n a **region of the symmetric spectrum of odd divisors for n**. Abusing terminology we will use the shorter terms **spectrum** and **region** in the sequel whenever possible.

THEOREM 2:

The following statements are equivalent:
(a) The spectrum for a number n contains an even number of regions.
(b) $w_{n,r_n} = 0$, i.e., #(odd divisors < r_n) = #(odd divisors > r_n)

PROOF:

(a) ⇒ (b): The regions in the first half of the spectrum for number n must end when the symmetric Dyck path comes to its middle on the diagonal. Therefore, $w_{n,r_n} = 0$, and for 1 ≤ k ≤ r_n there must be as many odd indices k as there are even indices k for which $d_{n,k} = 1$. Since by Lemma 1(e) an even index k for which $d_{n,k} = 1$ indicates a matched odd divisor greater than r_n, odd divisors of n are either smaller or larger than r_n and the equation follows.
(b) ⇒ (a): If $w_{n,r_n} = 0$ then the (n-1)-st and n-th symmetric Dyck path meet at their diagonal point and the last region ends by that point or earlier along the n-th symmetric Dyck path. By the symmetry of the Dyck paths around the diagonal the total number of regions is even.
■

The sums of numbers in each region for n are the areas of the regions except for the central region - when there is an odd number of regions - where the width $\overline{w}_{n,r_n}$ at the center point is counted in both legs of the Dyck paths meeting at the diagonal and needs to be subtracted. The sequence of areas of the regions for n form the n-th row in irregular triangle A237270.

DEFINITION 10: Sequence C (A237271):

We call the number of regions in the spectrum $s_n = (s_{n,1}, \ldots, s_{n,2 \times r_n})$ for n the **symmetric count for n** and denote it by c_n.

DEFINITION 11: Sequence V

The sequence of numbers $v_n = \sum_{i=1}^{2 r_n} s_{n,k} - \overline{w}_{n,r_n}$, for all n ≥ 1, is the sum of the areas of the regions between the n-th and (n-1)-st Dyck paths. We call v_n the **symmetric divisor sum for n**.

CONJECTURE

$v_n = \sigma(n)$, for all n ≥ 1, where $\sigma(n)$ is the **sum of all divisors of n** (A000203).

So far I verified the truth of the conjecture computationally for all n ≤ 1,000,000 (see also A241561)

We use $\sigma_0(n)$ to denote the **number of divisors of n**.

Numbers with Even or Odd Symmetric Counts or Symmetric Width One

In this section we characterize sequences of numbers n for which c_n is even or odd, respectively, those for which $w_n = 1$, and those satisfying both properties. All five different sequences are in OEIS, but the theorems in this section will give new characterizations for some and will show that two sequences have duplicate sequences in OEIS.

DEFINITION 12:

A divisor d of n ∈ ℕ is called a **middle divisor of n** if $\sqrt{n/2} \le d < \sqrt{2n}$.

LEMMA 4:

Let $n = 2^m \times q > 1$ where $m \ge 0$ and q is odd. Suppose that c_n is even, i.e., n ∈ A241561.

(a) There are odd numbers d, e ∈ ℕ such that $q = d \times e$, $2^m \times d$ is the largest divisor of n less than r_n, and e is the smallest divisor of n greater than r_n.

(b) Number n has no middle divisors, i.e., n ∈ A071561.

PROOF:

Since the spectrum for n has an even number of regions and since 1 is an odd divisor of n, by Lemma 1(e) there is at least one odd divisor of n larger than r_n. Let e be the smallest odd divisor greater than r_n and $d \times e = q$. Then $2^m \times d$ is a divisor of n and $2^{m+1} \times d \le r_n$. Suppose that for some odd divisor x of n and for some $0 \le k < m$ we have $2^m \times d < 2^k \times x \le r_n$. Since $x \le r_n$ there is an odd divisor $y > r_n$ such that $q = x \times y$; then y is represented by $2^{m+1} \times x \le r_n$. Therefore, $2^m \times x$ also is a divisor of n less than r_n implying $d < x$. Finally, since $d \times e = q = x \times y$ we get $r_n < y = \frac{d}{x} \times e < e$ which contradicts minimality of e so that $2^m \times d$ is the largest divisor less than or equal to r_n.

Let $q = d \times e$ and let $2^m \times d$ and e be the two extremal divisors of n from part (a). Lemma 1(b) states $\sqrt{n/2} \le r_n < \sqrt{2n}$ and we have the equivalence:

$$2^m \times d < \sqrt{n/2} \iff 2^{2m+1} \times d^2 < 2^m \times d \times e \iff 2^{m+1} \times d \times e < e^2 \iff \sqrt{2n} < e.$$

Therefore, n has no middle divisors.

■

LEMMA 5:

Let $n = 2^m \times q$ where $m \ge 0$ and q is odd, and suppose that n has no middle divisors, i.e., n ∈ A071561. Then c_n is even, i.e., n ∈ A241561.

PROOF:

Assume that x is the largest divisor of n satisfying $x < \sqrt{n/2}$ and that y is the smallest divisor of n satisfying $\sqrt{2n} \le y$. If $x = 2^k \times d$ with $0 \le k < m$ and d|q, then divisor $2 \times x \ge \sqrt{n/2}$ because of the maximality of x and since $\sqrt{n/2} \le 2 \times x < 2 \times \sqrt{n/2} = \sqrt{2n}$ it would be a middle divisor, contradicting the assumption. Therefore, $x = 2^m \times d$ with d odd. Similarly, if $y = 2 \times z$ for some divisor z of n, then divisor $z < \sqrt{2n}$ because of the minimality of y and since $\sqrt{2n} > z = \frac{y}{2} \ge \frac{\sqrt{2n}}{2} = \sqrt{n/2}$ it would be a middle divisor, contradicting the assumption. Therefore, y is odd.

Now there are odd numbers e and z such that $d \times e = q = z \times y$. If $e < y$ then $e < \sqrt{n/2}$ since there are no middle divisors and y is minimal. However $n = 2^m \times q = 2^m \times d \times e < 2^m \times d \times y < \sqrt{n/2} \times \sqrt{n/2} = n/2$ is a contradiction so that $e \ge y$. The assumption $z > d$ leads to the contradiction $n = 2^m \times q = 2^m \times z \times y > 2^m \times d \times y > \sqrt{2n} \times \sqrt{2n} = 2 \times n$ so that $z \le d$. Therefore, $1 \le \frac{d}{z} = \frac{y}{e} \le 1$ implies d = e and z = y. In other words, the largest odd divisor of n less than r_n is paired with the smallest odd divisor of n greater than r_n, i.e., the number of odd divisors less than r_n equals the number of odd divisors greater than r_n so that c_n is even by Theorem 2.

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These two Lemmas together with the fact that A071561 and A071562 are complementary establish the following equivalences.

THEOREM 3:

For every number $n \in \mathbb{N}$:
(a) c_n is even $\Leftrightarrow$ $n \in$ A071561 $\Leftrightarrow$ $n \in$ A241561 $\Leftrightarrow$ n has no middle divisors
(b) c_n is odd $\Leftrightarrow$ $n \in$ A071562 $\Leftrightarrow$ $n \in$ A241562 $\Leftrightarrow$ n has at least one middle divisor

We include the next two theorems in order to have a comprehensive list of known properties for sequences A241008 and A241010 in this note.

THEOREM 4:

For every number $n \in \mathbb{N}$:
$w_n = 1$ $\Leftrightarrow$ $n \in$ A174905 $\Leftrightarrow$ $n \in$ A241008 $\cup$ A241010
$\Leftrightarrow$ no two divisors p and q of n satisfy $p < q < 2 \times p$.

PROOF:

See the link named “Proof that this sequence equals union of A241008 and A241010”
https://oeis.org/A174905/a174905.pdf
in the LINKS section of A174905 in OEIS.
■

THEOREM 5:

For every number $n \in \mathbb{N}$:
c_n is odd & $w_n = 1$ $\Leftrightarrow$ $n \in$ A241010
$\Leftrightarrow$ $n = 2^{k-1} \times p^{2h}$, where $k > 0$, $p \geq 3$ is prime, $h \geq 0$, $2^k < p$ and $p^h \leq r_n < 2^k \times p^h$.

In this case there are $2 \times h + 1$ regions in the spectrum for n of respective sizes
$\frac{1}{2} \times \left(2^k - 1\right) \times \left(p^j + p^{2h-j}\right)$, $0 \leq j \leq 2 \times h$.
Specifically, the center region has size $\left(2^k - 1\right) \times p^h$ and $v_n = \sigma(n) = \left(2^k - 1\right) \times \sum_{j=0}^{2h} p^j$.

PROOF:

See the link named “Proof of characterization theorem”
https://oeis.org/A241010/a241010.pdf
in the LINKS section of A241010 in OEIS.
■

LEMMA 6:

Let $n = 2^m \times q = 2^m \times \prod_{i=1}^{k} p_i^{e_i}$ with $m \geq 0$, $k > 0$, $2 < p_1 < \ldots < p_k$ primes, and $e_i \in \mathbb{N}$, $e_i \geq 1$, for all $1 \leq i \leq k$, be the prime factorization of n. Suppose that at least one of e_i, $1 \leq i \leq k$, is odd and that for any two odd divisors $f < g$ of n, $2^{m+1} \times f < g$. Then $c_n = \sigma_0(q)$ is even and $w_n = 1$.

PROOF:

Since at least one of e_i, $1 \leq i \leq k$, is odd we get $\sigma_0(q) = \sigma_0(\prod_{i=1}^{k} p_i^{e_i}) = \prod_{i=1}^{k} (e_i + 1)$ is even. Suppose that the odd divisors of n are $1 = d_1 < \ldots < d_x < d_{x+1} < \ldots < d_{2x} = q$ where $2 \times x = \sigma_0(q)$. Then $d_y \times d_{2x+1-y} = q$, for all $1 \leq y \leq x$. By Lemma 1(e) the odd divisors d_{2x+1-y}, $1 \leq y \leq x$, are represented by 1’s in positions $2^{m+1} \times d_y$. Therefore, the condition $2^{m+1} \times f < g$ for any two odd divisors implies that 1’s in odd and even positions alternate in the n-th row of irregular triangle A237048, i.e., and $w_n = 1$.
■

LEMMA 7:

Let n = $2^m \times$ q = $2^m \times \prod_{i=1}^{k} p_i^{e_i}$ with m ≥ 0, k > 0, 2 < p_1 < … < p_k primes, and $e_i \in \mathbb{N}$, $e_i \geq 1$, for all 1 ≤ i ≤ k, be the prime factorization of n. If $c_n = \sigma_0(q)$ is even and w_n = 1 then k > 0 and at least one of e_i, 1 ≤ i ≤ k, is odd, and for any two odd divisors f < g of n, $2^{m+1} \times$ f < g.

PROOF:

Since $c_n = \sigma_0(q)$ is even n cannot equal 2^m, but must have at least one odd divisor greater than 1, i.e., k > 0. Furthermore, since there is an even number of 1's in the n-th row of irregular triangle A237048 the prime factorization of n must contain at least one odd prime number to an odd power. Since w_n = 1 the positions of the odd divisors d_i, 1 ≤ i ≤ $\sigma_0(q)$ = 2×x, represented by 1's in the n-th row of irregular triangle A237048 alternate between odd and even positions, i.e., $1 = d_0 < 2^{m+1} < d_1 < 2^{m+1} \times d_1 < d_2 < \ldots < d_x < 2^{m+1} \times d_x \leq r_n$. This chain of inequalities holds for all odd divisors since for $d_i \times d_{2\alpha+1-i} = d_{i+1} \times d_{2\alpha-1} = q$ we get $d_{2\alpha-1} < d_{2\alpha+1-i}$ so that $2^{m+1} \times d_{2\alpha-i} = \frac{2^{m+1} \times d_i}{d_{i+1}} \times d_{2\alpha+1-i} < d_{2\alpha+1-i}$.

■

THEOREM 6:

For every number n ∈ ℕ with prime factorization n = $2^m \times$ q = $2^m \times \prod_{i=1}^{k} p_i^{e_i}$ with m ≥ 0, k > 0, 2 < p_1 < … < p_k primes, and $e_i \in \mathbb{N}$, $e_i \geq 1$, for all 1 ≤ i ≤ k:
c_n is even & w_n = 1 ⇔ n ∈ A241008
⇔ k > 0, at least one of e_i, 1 ≤ i ≤ k, is odd, and for any two odd divisors f < g of n, $2^{m+1} \times$ f < g.

As in the proof above, let the odd divisors of n be $1 = d_1 < \ldots < d_x < d_{x+1} < \ldots < d_{2x} = q$, where 2×x = $\sigma_0(q)$. The z-th region of n has area $a_z = \frac{1}{2} \times (2^{m+1} - 1) \times (d_z + d_{2x+1-z})$, for 1 ≤ z ≤ 2×x, so that in this case $v_n = \sum_{z=1}^{2x} a_z = \sum_{z=1}^{2x} \frac{1}{2} \times (2^{m+1} - 1) \times (d_z + d_{2x+1-z}) = (2^{m+1} - 1) \times \sum_{z=1}^{x} (d_z + d_{2x+1-z}) = \sigma(n)$.

PROOF:

The equivalences follow from Lemmas 6 & 7. In order to verify the formula for the areas a_z, 1 ≤ z ≤ 2×x, we establish the following identities for the n-th row of irregular triangle F (A237591) that together show $v_n = \sigma(n)$ in this case:

(i) $f_{n, 2^{m+1}} = f_{n-1, 2^{m+1}} + 1 = \frac{q-1}{2} - 2^m + 1$

(ii) $f_{n, 2^{m+1} \times d_z} = f_{n-1, 2^{m+1} \times d_z} + 1 = \frac{1}{2} \times \left(\frac{q}{d_z} - 1\right) - 2^m \times d_z + 1$

(iii) $f_{n, d_z} = f_{n-1, d_z} + 1 = 2^m \times \frac{q}{d_z} - \frac{1}{2}(d_z + 1) + 1$

(iv) $f_{n, d_z} - f_{n, 2^{m+1} \times d_z} = \frac{1}{2} \times (2^{m+1} - 1) \times (d_z + d_{2x+1-z})$

(v) $f_{n,k} = f_{n-1,k}$, for all $1 \leq k \leq r_n$ with $k \neq d_z, 2^{m+1} \times d_z$,

Formulas (i) - (iv) are straightforward calculations. For (v) we argue as follows.
Let n = u×k + v with 0 ≤ v < k. Then
$f_{n,k} = \left\lceil \frac{u \times k + v + 1}{k} - \frac{k+1}{2} \right\rceil = u + \left\lceil \frac{v+1}{k} - \frac{k+1}{2} \right\rceil$ and $f_{n-1,k} = u + \left\lceil \frac{v}{k} - \frac{k+1}{2} \right\rceil$.
If k is odd and $k \neq d_z$ for any 1 ≤ z ≤ x then $\left\lceil \frac{v+1}{k} \right\rceil = \left\lceil \frac{v}{k} \right\rceil = 1$.
If k is even and $k \neq 2^{m+1} \times d_z$ for any 1 ≤ z ≤ x then
$f_{n,k} = u - \frac{k}{2} + \left\lceil \frac{v+1}{k} - \frac{1}{2} \right\rceil$ and $f_{n-1,k} = u - \frac{k}{2} + \left\lceil \frac{v}{k} - \frac{1}{2} \right\rceil$.
Case $0 \leq v < \frac{k}{2}$:
$\left\lceil \frac{v+1}{k} - \frac{1}{2} \right\rceil = 0 = \left\lceil \frac{v}{k} - \frac{1}{2} \right\rceil$ since 2×v < k and k even imply 2×v + 2 ≤ k.
Case $\frac{k}{2} < v < k$:
$\left\lceil \frac{v+1}{k} - \frac{1}{2} \right\rceil = 1 = \left\lceil \frac{v}{k} - \frac{1}{2} \right\rceil$ since 0 < 2×v - k.

Case $\frac{k}{2} = v$:

In this case $n = u \times k + v = u \times k + \frac{k}{2} = \frac{k}{2} \times (2 \times u + 1)$ so that $2 \times n = 2^{m+1} \times q = (2 \times u + 1) \times k$.

This implies that $2^{m+1} | k$ and $k = 2^{m+1} \times d_z$, for some z, contradicting the assumption on k.

■