Proof of conjectured formula for A068082

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The conjectured terms $f(n) = (169 \cdot 9^{n-3} - 1)/8 = x(x+1)/2$ where $x = (13 \cdot 3^{n-3} - 1)/2$, so these are triangular for $n \geq 3$, and $f(n+1) = 9f(n) + 1$. Thus all we need to do is verify that $kf(n) + 1$ is not a triangular number, i.e. $8kf(n) + 9$ is not a square, for $k = 0, 1, \ldots, 8$.

Note that $8kf(n) + 9 \equiv -k \bmod 9$ if $n \geq 4$, and this is not a square mod 9 unless $k = 5$ or $k = 8$.

For $k = 5$, $40f(n) + 9 = 845 \cdot 9^{n-3} + 4$. If this is y^2, i.e. $845 \cdot 9^{n-3} = y^2 - 4 = (y-2)(y+2)$. Now only one of $y - 2$ and $y + 2$ can be divisible by 9, so one of these is divisible by (and thus $\geq$) 9^{n-3}, and the other divides (and thus $\leq$) 845. If $n \geq 7$, $9^{n-3} > 845 + 4$, so this is impossible.

Similarly, for $k = 8$, $64f(n) + 9 = 1352 \cdot 9^{n-3} + 1$. If this is y^2, i.e., $1352 \cdot 9^{n-3} = (y - 1)(y + 1)$, and again this is impossible for $n \geq 7$ as $9^{n-3} > 1352 + 2$.