A signed Dirichlet product of arithmetical functions

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We investigate a twisted version of the Dirichlet product of arithmetical functions and show it has similar properties to the standard Dirichlet product. We give some examples of sequences in the OEIS that can be described in terms of this twisted Dirichlet product. In Section 6 we briefly mention further twisted versions of the Dirichlet product.

1. The signed Dirichlet product Given two arithmetical functions $f(n)$ and $g(n)$ (functions from the positive integers to the complex numbers) their Dirichlet product (or Dirichlet convolution) $f * g$ is the arithmetical function defined by the divisor sum

$$\begin{aligned} (f * g)(n) &= \sum_{d|n} f(d) g\left(\frac{n}{d}\right) \\ &= \sum_{ab=n} f(a) g(b), \end{aligned} \tag{1}$$

where the latter sum is over all positive integers a and b such that $ab = n$.

An equivalent definition can be stated in terms of products of formal Dirichlet series. The Dirichlet product $f * g$ is the arithmetical function determined by the formula

$$\sum_{n\geq 1} \frac{(f * g)(n)}{n^s} = \left(\sum_{n\geq 1} \frac{f(n)}{n^s}\right) \left(\sum_{n\geq 1} \frac{g(n)}{n^s}\right). \tag{2}$$

Dirichlet convolution is commutative, associative and distributes over addition. See, for example, [Apo, Chapter 2].

Definition We define the *signed Dirichlet product* $f \star g$ of f and g to be the arithmetical function determined by the formula

$$\sum_{n\geq 1} \frac{(-1)^{n+1} (f \star g)(n)}{n^s} = \left(\sum_{n\geq 1} \frac{(-1)^{n+1} f(n)}{n^s}\right) \left(\sum_{n\geq 1} \frac{(-1)^{n+1} g(n)}{n^s}\right). \tag{3}$$

Comparing the coefficients of n^{-s} on both sides of the equation gives

$$(-1)^{n+1} (f \star g)(n) = \sum_{d|n} (-1)^{d+\frac{n}{d}} f(d) g\left(\frac{n}{d}\right), \tag{4}$$

leading to

$$(f \star g)(n) = \sum_{ab=n} (-1)^{(1+a)(1+b)} f(a) g(b), \tag{5}$$

where the sum is over all positive integers a and b such that $ab = n$.

Equation (5) can also be written as

$$(f \star g)(n) \quad = \quad \sum_{\text{odd } d|n} f(d)g\left(\frac{n}{d}\right) \; - \sum_{\text{even } d|n} (-1)^{\frac{n}{d}} f(d)g\left(\frac{n}{d}\right). \qquad (6)$$

Let $s(n) = (-1)^{n+1}$. It will be convenient to associate with an arithmetical function $f(n)$ the function $\bar{f}(n)$ given by

$$\bar{f}(n) = s(n)f(n).$$

Clearly, $\bar{\bar{f}}(n) = f(n)$. Our first result relates the Dirichlet product and the signed Dirichlet product.

Theorem 1.1 *Let $f(n)$ and $g(n)$ be arithmetical functions. Then*

$$(f \star g)(n) = \overline{(\bar{f} * \bar{g})}(n).$$

Proof. Immediate by comparing (2) and (3). ■

By means of Theorem 1.1, the algebraic properties of the $\star$ product can be deduced from the corresponding properties of the Dirichlet product. In what follows we closely parallel the treatment of the Dirichlet product in Apostol [Apo, Chapter 2].

Theorem 1.2 *The signed Dirichlet product operation is commutative and associative and distributes over addition. That is, for any arithmetical functions f, g and h, we have*

$$f \star g = g \star f$$

$$(f \star g) \star h = f \star (g \star h)$$

$$f \star (g + h) = f \star g + f \star h.$$

Proof. The commutativity and the distributivity property are both evident from the definition (3) of the signed Dirichlet product. As for associativity, we have by Theorem 1.1

$$\begin{aligned} (f \star g) \star h &= \overline{(\bar{f} * \bar{g})} \star h \\ &= \overline{\left(\overline{\overline{(\bar{f} * \bar{g})}} * \bar{h}\right)} \\ &= \overline{((\bar{f} * \bar{g}) * \bar{h})}. \end{aligned} \qquad (7)$$

Again by Theorem 1.1 we have

$$\begin{aligned} f \star (g \star h) &= \overline{\left(\bar{f} * \overline{(g \star h)}\right)} \\ &= \overline{\left(\bar{f} * \overline{\overline{(\bar{g} * \bar{h})}}\right)} \\ &= \overline{(\bar{f} * (\bar{g} * \bar{h}))}. \end{aligned} \tag{8}$$

The associativity of the $\star$ product now follows by comparing (7) and (8) and using the associativity of the Dirichlet product $*$. ■

2. Dirichlet inverses The multiplicative identity (multiplicative unit) for both the Dirichlet product and the signed Dirichlet product is the arithmetical function ε given by

$$\varepsilon(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Clearly, $\bar{\varepsilon}(n) = \varepsilon(n)$. Apostol [Apo, Theorem 2.8] shows that if f is an arithmetical function with $f(1) \neq 0$ there is a unique arithmetical function f^{-1}, called the Dirichlet inverse of f, such that

$$f * f^{-1} = f^{-1} * f = \varepsilon.$$

Theorem 2.1 *If f is an arithmetical function with $f(1) \neq 0$, then there is a unique arithmetical function $f^{\langle -1 \rangle}$ such that*

$$f \star f^{\langle -1 \rangle} = f^{\langle -1 \rangle} \star f = \varepsilon,$$

given by

$$f^{\langle -1 \rangle} = \overline{(\bar{f})^{-1}}.$$

We refer to $f^{\langle -1 \rangle}$ as the signed Dirichlet inverse of f.

PROOF. Suppose g is an arithmetic function such that $f \star g = \varepsilon$. By Theorem 1.1, we have $\overline{(\bar{f} * \bar{g})} = \varepsilon$, or equivalently, $\bar{f} * \bar{g} = \epsilon$. So $\bar{g}$ is the unique Dirichlet inverse of $\bar{f}$ and hence $g = \overline{(\bar{f})^{-1}}$, and is determined uniquely. ■

3, Multiplicative functions and the signed Dirichlet product

A multiplicative function is a nonzero arithmetic function f with the property $f(nm) = f(n)f(m)$ whenever $gcd(n, m) = 1$. A multiplicative function f is called completely mutiplicative if $f(nm) = f(n)f(m)$ for all n, m.

The Dirichlet product of two multiplicative functions is multiplicative [Apo, Theorem 2.14] and the Dirichlet inverse of a multiplicative function is multiplicative [Apo, Theorem 2.16]. Next we prove the corresponding results for the signed Dirichlet product. First we need the following simple result.

Lemma 3.1 *If $f(n)$ is a multiplicative function then $\bar{f}(n)$ is a multiplicative function.*

PROOF. We show the arithmetic function $s(n) := (-1)^{n+1}$ is multiplicative. The result will then follow since $\bar{f}(n) = s(n)f(n)$, and the product of two multiplicative functions is again multiplicative. Let n and m be coprime natural numbers. Then

$$\begin{aligned} s(n)s(m) &= (-1)^{n+1+m+1} \\ &= (-1)^{n+m+nm+1}(-1)^{nm+1} \\ &= (-1)^{(n+1)(m+1)}s(nm) \\ &= s(nm), \end{aligned}$$

since when n and m are coprime at least one of n and m must be odd and hence $(-1)^{(n+1)(m+1)} = 1$. ■

Theorem 3.1 *If f and g are multiplicative then the signed Dirichlet product function $f \star g$ is also multiplicative.*

PROOF. By Theorem 1.1, the signed Dirichlet product function $f \star g = \overline{(\bar{f} * \bar{g})}$. By the Lemma, both functions $\bar{f}$ and $\bar{g}$ are multiplicative. Hence their Dirichlet product $\bar{f} * \bar{g}$ is multiplicative [Apo, Theorem 2.14]. Another application of the Lemma shows that the function $\overline{(\bar{f} * \bar{g})}$ is multiplicative, and hence $f \star g$ is mutiplicative. ■

Theorem 3.2 If both g and $f \star g$ are multiplicative then f is multiplicative.

PROOF. Since g is multiplicative, so is $\bar{g}$ by the Lemma. By Theorem 1.1 and the assumption, $f \star g = \overline{(\bar{f} * \bar{g})}$ is multiplicative. Hence by the Lemma, $\bar{f} * \bar{g}$ is multiplcative. Thus both $\bar{g}$ and the Dirichlet product $\bar{f} * \bar{g}$ are multiplicative. It follows from [Apo, Theorem 2.15] that $\bar{f}$ is multiplicative and hence, by a final application of the Lemma, f is multiplicative. ■

Theorem 3.3 *If f is a multiplicative function then the signed Dirichlet inverse $f^{\langle -1 \rangle}$ is also multiplicative.*

PROOF. Immediate from Theorem 3.2, since both f and $f \star f^{\langle -1 \rangle} = \epsilon$ are multiplicative. ■

4. Lambert type series Given an arithmetical function $a(n)$, consider the following variant of a Lambert series:

$$L(q) = \sum_{n=1}^{\infty} a(n) \frac{q^n}{1 + (-q)^n}.$$

Formally expanding the denominators gives the formal power series expansion

$$\begin{aligned} L(q) &= \sum_{m \text{ odd}} a(m) \sum_{k=1}^{\infty} q^{mk} + \sum_{m \text{ even}} a(m) \sum_{k=1}^{\infty} (-1)^{k+1} q^{mk} \\ &= \sum_{n=1}^{\infty} b(n) q^n. \end{aligned}$$

Comparing the coefficients of q^n on both sides of the equation gives the coefficient $b(n)$ of the power series $L(q)$ as

$$b(n) = \sum_{\text{odd } d \mid n} a(d) + \sum_{\text{even } d \mid n} (-1)^{\frac{n}{d}+1} a(d). \tag{9}$$

By (6), we see that the arithmetic function $b(n)$ is simply the signed Dirichlet product $a \star 1$. One immediate consequence by Theorem 3.1, is that if $a(n)$ is multiplicative then so is $b(n)$.

Here are some examples of sequences of the form $a \star 1$ already in the OEIS database. $1(n)$ denotes the constant function $1(n) = 1$ (completely multiplicative): $\mathrm{Id}(n)$ denotes the identity function $\mathrm{Id}(n) = n$ (completely multiplicative). More generally, $\mathrm{Id}_k(n)$ denotes the k^{th} power function defined by $\mathrm{Id}_k(n) = n^k$ (also completely multiplicative).

Example 1. $\phi \star 1$, where ϕ denotes the Euler totient function. The expansion of the Lambert series associated with the function ϕ begins

$$\sum_{n=1}^{\infty} \phi(n) \frac{q^n}{1 + (-q)^n} = q + 2q^2 + 3q^3 + 2q^4 + 5q^5 + 6q^6 + 7q^7 + 2q^8 + \cdots$$

This is the ordinary generating function for A259445. Since the Euler totient function is multiplicative, the coefficients $b(n) = (\phi \star 1)(n)$ of the power series also form a multiplicative function. The value of $b(n)$ is thus determined by the values the function b takes at the prime powers. Using the well-known result $\sum_{d|n} \phi(d) = n$, we see from (9) with $a = \phi$ that $b(n) = n$ when n is odd, so in particular $b(p^n) = p^n$ for odd primes p. An easy induction argument using (9) shows that $b(2^n) = 2$ for $n = 1, 2, 3, \ldots$.

Example 2. $1\star 1$. The expansion of the Lambert series begins

$$\sum_{n=1}^{\infty} \frac{q^n}{1+(-q)^n} = q + 2q^2 + 2q^3 + q^4 + 2q^5 + 4q^6 + 2q^7 + 3q^9 + \cdots.$$

This is the ordinary generating function for $(-1)^{n+1}$ A228441. The power series coefficients are given by the formula
$b(n) = (1 \star 1)(n) = \sum_{d|n} (-1)^{(1+d)\left(1+\frac{n}{d}\right)}$. The arithmetic function $b(n)$ is multiplicative by Theorem 3.1.

Example 3. $\mathrm{Id}\star 1$. The expansion of the Lambert series begins

$$\sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n} = q + 3q^2 + 4q^3 + 3q^4 + 6q^5 + 12q^6 + 8q^7 + 3q^8 + \cdots.$$

This is the ordinary generating function for A046897. The power series coefficients are given by $b(n) = (\mathrm{Id} \star 1)(n) = \sum_{d|n} (-1)^{(1+d)\left(1+\frac{n}{d}\right)} d$. Again the arithmetic function $b(n)$ is multiplicative by Theorem 3.1.

More generally, the signed Dirichlet products $\mathrm{Id}_k \star 1$ for $k = 2$ through $k = 9$ are given by $(-1)^{n+1}$A321558(n) through $(-1)^{n+1}$A321565(n). For the cases $k = 10, 11, 12$ see A321807 , A321808 and A321809.

5. Möbius inversion

We now consider the analogue of the Möbius inversion formula for the signed Dirichlet product. Recall the Möbius function $\mu(n)$ is the Dirichlet inverse of the constant function 1.

$$\mu * 1 = \varepsilon.$$

Given arithmetic functions f and g, the Möbius inversion formula states

$$f = g * 1 \quad \Longleftrightarrow \quad g = f * \mu.$$

In terms of divisor sums this is equivalent to the inversion formula

$$f(n) = \sum_{d|n} g(d) \quad \Longleftrightarrow \quad g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d).$$

We denote the signed Dirichlet inverse of the constant function 1 by $\widetilde{\mu}(n)$:

$$\widetilde{\mu} \times 1 = \varepsilon$$

or

$$\widetilde{\mu} = 1^{\langle -1 \rangle}.$$

Define the alternating zeta function $\zeta_A(s)$ as the (formal) Dirichlet series $\zeta_A(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$. It follows from (3) applied to the identity $\widetilde{\mu} \star 1 = \varepsilon$ that the Dirichlet series associated with the function $(-1)^{n+1}\widetilde{\mu}(n)$ is the reciprocal of the alternating zeta function, that is,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\widetilde{\mu}(n)}{n^s} = \frac{1}{\zeta_A(s)}.$$

Using $\zeta_A(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \left(1 - \frac{2}{2^s}\right)\zeta(s)$ and $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ we see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\widetilde{\mu}(n)}{n^s} = \left(1 + \frac{2}{2^s} + \frac{4}{2^{2s}} + \cdots\right) \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}.$$

Equating the coefficients of n^{-s} on each side, we find that if 2^k is the largest power of 2 dividing n then

$$\widetilde{\mu}(n) \quad = \quad (-1)^{n+1}\left(\mu(n) + 2\mu\left(\frac{n}{2}\right) + \cdots + 2^k\mu\left(\frac{n}{2^k}\right)\right). \tag{10}$$

The function $\widetilde{\mu}(n)$ is multiplicative by Theorem 3.3 with $\widetilde{\mu}(1) = 1$. Hence the values of $\widetilde{\mu}(n)$ are determined by the values $\widetilde{\mu}$ takes on prime powers. If n is odd it follows from (10) that $\widetilde{\mu}(n) = \mu(n)$, so for odd prime p, $\widetilde{\mu}(p^m) = \mu(p^m)$, which has the value -1 when $m = 1$, otherwise equals 0.

In the case that n is even we can use the fact that $\mu(n) = 0$ if n is not square free to shorten the sum in (10) to two terms:

$$\begin{aligned}
\widetilde{\mu}(n) \quad &= \quad (-1)^{n+1}\left(2^{k-1}\mu\left(\frac{n}{2^{k-1}}\right) + 2^k\mu\left(\frac{n}{2^k}\right)\right) \\
&= \quad (-1)^{n+1}\left(2^{k-1}\mu\left(2\frac{n}{2^k}\right) + 2^k\mu\left(\frac{n}{2^k}\right)\right) \\
&= \quad (-1)^{n+1}2^{k-1}\mu\left(\frac{n}{2^k}\right),
\end{aligned}$$

since the Möbius function μ is multiplicative. In particular, we get for $m \geq 1$, $\widetilde{\mu}(2^m) = -2^{m-1}$.

The table below lists the first few values of $\widetilde{\mu}(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	...
$\widetilde{\mu}(n)$	1	-1	-1	-2	-1	1	-1	-4	0	1	-1	2	...

This is the sequence $(-1)^{n+1}$A067856(n).

For the signed Dirichlet product $\star$, the analogue of Möbius inversion reads

$$f = g \star 1 \Longleftrightarrow g = f \star \widetilde{\mu},$$

for arithmetic functions f, g. In terms of divisor sums this is equivalent to the inversion formula

$$f(n) = \sum_{d|n} (-1)^{(1+d)\left(1+\frac{n}{d}\right)} g(d) \Longleftrightarrow g(n) = \sum_{d|n} (-1)^{(1+d)\left(1+\frac{n}{d}\right)} \widetilde{\mu}\left(\frac{n}{d}\right) f(d), \tag{11}$$

or equivalently,

$$(-1)^{n+1} f(n) = \sum_{d|n} (-1)^{\left(d+\frac{n}{d}\right)} g(d) \Longleftrightarrow (-1)^{n+1} g(n) = \sum_{d|n} (-1)^{\left(d+\frac{n}{d}\right)} \widetilde{\mu}\left(\frac{n}{d}\right) f(d). \tag{12}$$

6. Further twisted Dirichlet products

Let now $s(n)$ be a **multiplicative** arithmetic function with $s(1) = 1$ and satisfying the property $s(n)^2 = 1$, so that $s(n) \in \{1, -1\}$. In the above we worked with the choice $s(n) = (-1)^{n+1}$. Let f, g be arithmetic functions. Define *the Dirichlet product*

$f \underset{s}{\star} g$ *of f and g twisted by the function s* to be the arithmetic function determined from

$$\sum_{n \geq 1} s(n) \frac{\left(f \underset{s}{\star} g\right)(n)}{n^s} = \left(\sum_{n \geq 1} \frac{s(n) f(n)}{n^s}\right) \left(\sum_{n \geq 1} \frac{s(n) g(n)}{n^s}\right). \tag{13}$$

This is equivalent to the divisor sum

$$s(n) \left(f \underset{s}{\star} g\right)(n) = \sum_{d|n} s(d) f(d) s\left(\frac{n}{d}\right) g\left(\frac{n}{d}\right). \tag{14}$$

Define $\bar{f}(n) = s(n) f(n)$ so that $\bar{\bar{f}}(n) = f(n)$. It is easy to see that the Dirichlet product twisted by the function s is related to the standard Dirichlet product by

$$\left(f \underset{s}{\star} g\right)(n) = \overline{(\bar{f} * \bar{g})}(n). \tag{15}$$

Exactly as in Section 2 and Section 3, it can be shown that the twisted Dirichlet product is commutative, associative and distributes over addition: the twisted product of multiplicative functions is multiplicative and a multiplicative arithmetic function f has a multiplicative inverse with respect to the twisted Dirichlet product (the function ε is still the multiplicative unit).

There is also a twisted verson of Möbius inversion. Define the twisted Möbius function μ_s to be the inverse of the constant function 1 under the twisted Dirichlet product:

$$\mu_s \underset{s}{\star} 1 = \varepsilon.$$

Then for arithmetic functions f and g the following inversion formula holds:

$$s(n)f(n) = \sum_{d|n} s(d)s\left(\frac{n}{d}\right)g(d) \iff s(n)g(n) = \sum_{d|n} s(d)s\left(\frac{n}{d}\right)\mu_s\left(\frac{n}{d}\right)f(d). \tag{16}$$

References

T. M. Apostol, *Introduction to Analytic Number Theory, Springer-Verlag 1976.*