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A058265: A geometric construction of the tribonacci constant with marked ruler and compass

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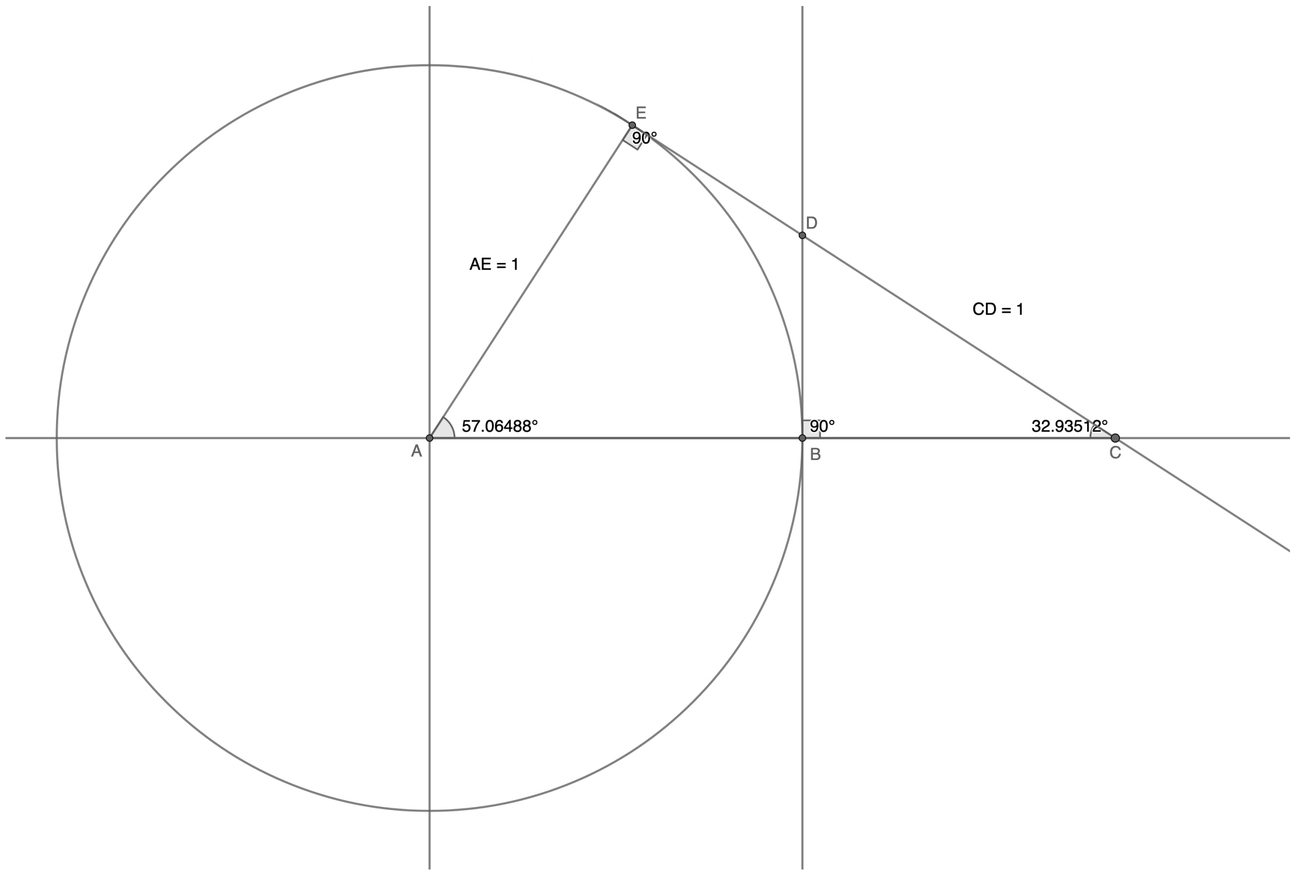
The tribonacci constant ('t', sometimes referred with Greek letters eta 'η' or tau 'τ') is related to the tribonacci numbers sequence (OEIS **A000073** : 0, 0, 1, 1, 2, 4, 7...) as the Golden Ratio is related to the Fibonacci sequence, and it is the only real solution to the equation $x^3 - x^2 - x - 1 = 0$ [1] [2].
Its decimal expansion, as appears in OEIS **A058265** , is:

1.839286755214161132551852564653286600424178746097592246778758639404203222O...

It also responds to this formula [3]:

$$t = 1/3 \left[1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right]$$

Thus, tribonacci constant is not a constructible number, with compass and straightedge: in ancient Greece, mathematician Nicomedes gave a method for construct any cubic root with neusis [4] [5], and geometers like Persian Omar Khayyam were capable of solve cubic equations with conics [6].
But, we present here a simple and apparently unpublished geometric construction of the tribonacci constant with marked ruler and compass (but not properly a neusis construction):



We first draw a unit circle with origin at A, and then a straight line, the vertical tangent passing through B: now, we must put the marked ruler against the circle as another tangent, while putting the marks for the lenght of the radius on the AB line and the tangent passing through B, creating point C, point D and point E. Now, let's call segment AC as t: so, we can see two equations for the angle BCD, $\sin(BCD) = 1 / t$ and $\cos(BCD) = (1 - t) / 1$, and, from $\sin^2 + \cos^2 = 1$, that leads to a quartic equation, $t^4 - 2t^3 + 1 = 0$, which has the next factorization:

$$(t - 1) \cdot (t^3 - t^2 - t - 1) = 0$$

This equation has two solutions: the first one, for $(t - 1)$, it will be evidently $t = 1$, that is, the tangent on the unit circle; and the other one, for $(t^3 - t^2 - t - 1)$, as we established at the beginning of this article, has only one real solution, that is, the tribonacci constant. *Quod erat demonstrandum*.

References

[1] John Sharp, *Beyond the Golden Section - the Golden tip of the iceberg*, https://archive.bridgesmathart.org/2000/bridges2000-87.pdf

[2] John Sharp, *Have you seen this number?*, https://www.jstor.org/stable/3620403

[3] H. Martyn Cundy, *Snubbing with and without Eta*, https://www.jstor.org/stable/3621469

[4] Arthur Baragar, *Constructions Using a Compass and Twice-Notched Straightedge*, http://baragar.faculty.unlv.edu//papers/monthly151-164.pdf

[5] Roger C. Alperin, *Trisections and Totally Real Origami*, https://www.jstor.org/stable/30037438

[6] Wolfdieter Lang, *A Geometrical Problem of Omar Khayyám and its Cubic*, https://www.itp.kit.edu/~wl/EISpub/A256099.pdf

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