

With regard to Nester (1999), this document contains:

1. notes and errata for Chapter 2 (Finite and periodic sequences);
2. the original text for Chapter 2 with approximate page numbers;
3. the original Bibliography; and
4. the original Appendix A.

Nester (1999). Mathematical investigations of some plant interaction designs. PhD thesis.  
University of Queensland.

## **Notes and errata for Chapter 2**

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**7 August 2017**

These notes and errata refer to Chapter 2 of my PhD thesis. They update the errata from 24 April 2017. I do not anticipate any further updates.

Nester (1999). Mathematical investigations of some plant interaction designs. PhD thesis. University of Queensland.

The page numbers below refer to the original thesis. They do not all exactly match this 2017 copy of the original thesis.

1. **VERY IMPORTANT:** The “equivalently” claim in the statement of Lemma 2.4 is sometimes wrong. The initial statement and proof of Lemma 2.4 are correct, but I did not offer any proof that the “equivalently” claim itself is correct! Fortunately, and contrary to earlier fears, I did not use the “equivalently” claim in any of my Maple scripts when calculating the numbers of sequences (words) of a given type. Consequently there is nothing to indicate that any of the sequences which I submitted to the OEIS contain errors.

Here are some further details: The equivalently claim is correct for all moduli between 2 and 13 inclusive. The claim sometimes fails for each of the following moduli: 14, 18, 20, 22, 24, 26, 28, 30, 31. When I wrote my thesis I cross-checked some of the output from my Maple scripts with the results from explicit combinatorial computer searches written in C. Obviously all of those cross-checks were done on sequences (words) of length less than 14. I have now done comprehensive computer searches for all I, R, C, D, H, E equivalences classes (refer to thesis for meaning of these) on binary sequences (words) of length up to 16 inclusive, and on ternary sequences (words) of length up to 14 inclusive. The results of these computer searches are in complete agreement with the corresponding values in Tables 1-48 of Appendix A. I also carried out miscellaneous computer searches for Tables 49-64 concerning palindromes. Again complete agreement for corresponding values.

2. Section 2.4 p.76: Replace “Then the partition of  $n$  is said” with “Then the partition of  $q$  is said”.
3. Section 2.5 p.78: Replace “ $E(d-1,d)=2$  for  $d>1$ ” with “ $E(d-1,d)=2$  for  $d>2$ ”.
4. Section 2.6.6 p.86: In the statement of Corollary 2.14 the arguments for the proponent should be “ $(x_1, \dots, x_p)$ ”.
5. Section 2.7 p.86: In the middle of the first paragraph, replace “Let  $f(d)$  denote” with “Let  $f(n)$  denote”.

## Chapter 2. Finite and periodic sequences

From the survey in Chapter 1, it is apparent that fixed sequences of plant types form the basic building blocks for many plant interaction designs, e.g. the string designs and sequential designs. It is therefore natural to inquire into certain mathematical properties of sequences. Some of the computer software which was developed during these inquiries was used to obtain representatives of certain equivalence classes and these, in turn, are used in Chapter 3 for the investigation of sequential arrays.

Parts of Nester (1997) have been incorporated in this chapter.

### 2.1 Introduction to finite and periodic sequences

Consider an alphabet  $A_q$  which has exactly  $q$  different characters. A finite sequence, or word, of length  $n$  has form  $(s_1, s_2, \dots, s_n)$  where each  $s_i$  is an element of  $A_q$ . Occasionally we shall use  $s_1s_2\dots s_n$  to denote a finite sequence. A periodic sequence with period  $n$  has form  $(\dots, s_{-2}, s_{-1}, s_0, s_1, s_2, \dots)$  where all  $s_i$  are elements of  $A_q$  and where  $s_i = s_{i+n}$  for all  $i$ . Because of its repetitive nature, a periodic sequence can be represented by any of the following finite sequences, each of length  $n$ :  $(s_0, s_1, \dots, s_{n-1})$ ,  $(s_1, s_2, \dots, s_n)$ ,  $\dots$ ,  $(s_{n-1}, s_n, \dots, s_{2n-2})$ . We shall mostly use  $(s_1, s_2, \dots, s_n)$  as the finite sequence representation of a periodic sequence. Periodic sequences with period  $n$  are frequently characterized as necklaces with  $n$  beads.

No matter whether we are dealing with finite or periodic sequences we shall sometimes use  $S_i$  to denote the  $i^{\text{th}}$  element of sequence  $S$ .

A trivial sequence is one whose elements are all identical.

If  $S = (s_1, s_2, \dots, s_n)$  and  $T = (t_1, t_2, \dots, t_m)$  are finite sequences then sequences S and T are said to be equal if both  $m=n$  and  $s_i = t_i$  for all  $i$ . The concatenation ST of finite sequences S and T is defined to be  $(s_1, s_2, \dots, s_n, t_1, t_2, \dots, t_m)$ . The reverse of the finite sequence S is defined to be  $R(S) = (s_n, s_{n-1}, \dots, s_2, s_1)$ . Obviously

$$\begin{aligned} R(ST) &= R(s_1, s_2, \dots, s_n, t_1, t_2, \dots, t_m) \\ &= (t_m, \dots, t_1, s_n, \dots, s_1) \\ &= R(T)R(S). \end{aligned}$$

A primitive finite sequence is a finite sequence which is not the concatenation of  $k$  identical sequences each of length  $n/k$  for some  $k$  which divides  $n$ . For example,  $(0,1,0,0,2)$  is primitive but  $(1,1)$  and  $(0,1,0,1)$  are not primitive.

If  $S = (\dots, s_{-1}, s_0, s_1, \dots, s_n, s_{n+1}, \dots)$  is a periodic sequence with period  $n$  and  $T = (\dots, t_{-1}, t_0, t_1, \dots, t_m, t_{m+1}, \dots)$  is a periodic sequence with period  $m$  then sequences S and T are said to be equal if there is a fixed  $j$  such that  $s_i = t_{i+j}$  for all  $i$ . An important implication of this definition of equality is that for an arbitrary periodic sequence S it does not matter which element we designate as  $s_0$ .

The reverse of the periodic sequence S is defined to be  $R(S) = (\dots, s_2, s_1, s_0, s_{-1}, s_{-2}, \dots)$ .

If sequence S has period  $n$  then it also has period  $jn$  where  $j$  is any positive integer. A primitive periodic sequence is a periodic sequence for which the period  $n$  is the least positive  $m$  such that  $s_i = s_{i+m}$  for all  $i$ .

## 2.2 Transformations of sequences

We shall transform our sequences in two ways – either by permuting the positions in the sequences or by permuting the alphabetical characters which are allocated to sequence positions.

Let  $S = (s_1, s_2, \dots, s_n)$  be either a finite sequence or the finite sequence representation of a periodic sequence. All of our permutations of the positions of  $S$  will be linear transformations  $Q_{c,k}$  such that  $Q_{c,k}(S) = (s_{c+1k}, s_{c+2k}, \dots, s_{c+nk})$  with the subscripts reduced modulo  $n$  if necessary. Thus under the action of  $Q_{c,k}$ , element  $s_i$  is replaced by  $s_{c+ik}$ . Alternatively, the action could be interpreted as one of relocation rather than replacement. The actual interpretation used has no significant impact on the discussion below, provided we are consistent throughout. Also, because we are only interested in transformations  $Q_{c,k}$  which are permutations, we insist that  $k$  and  $n$  be relatively prime, symbolically  $(k,n)=1$ .

**Example 2.1:** Suppose  $S = (1, 2, 3, 4, 5)$ .  $S$  is special because  $s_i = i$  for all  $i$ , and  $S$  may be interpreted as a sequence of sequence positions. We find that

$Q_{2,2}(S) = (s_{2+1 \times 2}, s_{2+2 \times 2}, s_{2+3 \times 2}, s_{2+4 \times 2}, s_{2+5 \times 2}) = (s_4, s_1, s_3, s_5, s_2) = (4, 1, 3, 5, 2)$ . If we consider this as a permutation of sequence positions then we could write the permutation as

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix}$  which means that position 1 is replaced by position 4, position 2 is replaced by position 1, and so on. Similarly,

$Q_{3,1}(S) = (s_{3+1 \times 1}, s_{3+2 \times 1}, s_{3+3 \times 1}, s_{3+4 \times 1}, s_{3+5 \times 1}) = (s_4, s_5, s_1, s_2, s_3) = (4, 5, 1, 2, 3)$  with

corresponding permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$ .

We observe that  $Q_{c,k}(S)_i = s_{c+ik}$ , and  $Q_{0,1}$  is the identity permutation because

$Q_{0,1}(S)_i = s_{0+i1} = s_i$ . Because subscripts are reduced modulo  $n$ , we also note that if  $c_1 \equiv c_2 \pmod{n}$  and  $k_1 \equiv k_2 \pmod{n}$  then  $Q_{c_1, k_1}$  has the same effect as  $Q_{c_2, k_2}$ .

Suppose  $f$  and  $g$  are two functions. We denote function composition by  $f \circ g$ , the juxtaposition of the functions, and interpret it as being the combined effect of first applying  $g$ , and then applying  $f$  to the results obtained after applying  $g$ .

The following formula for function composition from Nester (1997) will be useful.

**Lemma 2.1.**  $Q_{c_2, k_2} Q_{c_1, k_1} = Q_{c_1 + k_1 c_2, k_1 k_2}$

*Proof:* Let  $S$  be an arbitrary sequence with elements  $s_i$ . Then

$$\begin{aligned} Q_{c_2, k_2} \{Q_{c_1, k_1}(S)\}_i &= Q_{c_1, k_1}(S)_{c_2 + k_2 i} \\ &= s_{c_1 + k_1(c_2 + k_2 i)} \\ &= s_{c_1 + k_1 c_2 + k_1 k_2 i} \\ &= Q_{c_1 + k_1 c_2, k_1 k_2}(S)_i. \end{aligned}$$

**Example 2.2:** Continuing example 2.1, the effect of applying  $Q_{2,2}$  and then  $Q_{3,1}$  is

$Q_{3,1} Q_{2,2} = Q_{2+2 \times 3, 1 \times 2} = Q_{3,2}$ , so that

$Q_{3,2}(S) = (s_{3+1 \times 2}, s_{3+2 \times 2}, s_{3+3 \times 2}, s_{3+4 \times 2}, s_{3+5 \times 2}) = (s_5, s_2, s_4, s_1, s_3) = (5, 2, 4, 1, 3)$ . Using the permutation notation, we place the permutation which is performed first on the right and obtain the composition of permutations as follows.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$ . Thus 1 is replaced by 4 in the second permutation, but 4 was replaced by 5 in the first permutation, with the net result that 1 is replaced by 5 in the composition.

We shall gather together the linear transformations into the following groups of permutations.

$I = \{Q_{0,1}\}$ . This is the identity permutation and leaves a sequence unchanged.

$C_n = \{Q_{c,1} \mid c \in \{0, 1, \dots, n-1\}\}$ . The permutation  $Q_{c,1}$  changes the sequence  $S = (s_1, s_2, \dots, s_n)$  to  $Q_{c,1}(S) = (s_{c+1}, s_{c+2}, \dots, s_{n-1}, s_n, s_1, \dots, s_c)$ , thereby replacing element  $s_i$  by  $s_{c+i}$  and shifting the sequence  $c$  steps to the left.  $C_n$  is the cyclic group of all possible cyclic shifts of finite sequences. It is obvious that translating a periodic sequence  $c$  places to the left is equivalent to performing a cyclic shift on its finite sequence representation, and that

this action leaves the periodic sequence unchanged. Translating a periodic sequence has the same effect as rotating a necklace within a plane.

$R_n = \{Q_{0,1}, Q_{1,-1}\}$ . This is called the group of reversals. As stated above, the first transformation is the identity permutation, and the second transformation forces reversal since

$$\begin{aligned} Q_{1,-1}(s_1, s_2, \dots, s_n) &= (s_{1-1}, s_{1-2}, \dots, s_{1-n}) \\ &= (s_0, s_{-1}, \dots, s_{-(n-1)}) \\ &= (s_{0+n}, s_{-1+n}, \dots, s_{-(n-1)+n}) \\ &= (s_n, s_{n-1}, \dots, s_1). \end{aligned}$$

If we think of a finite sequence as being the position numbers of beads in a necklace then reversal has the same effect as turning over the plane of a necklace and rotating if necessary. Much depends on which axis we use for flipping.

$H_n = \{Q_{0,k} \mid 1 \leq k < n, (n, k) = 1\}$ . I call these transformations step shifts. In Golomb (1967), step shifts were used in the investigation of the autocorrelation properties of sequences, and a  $k$ -step shift with  $k$  relatively prime to the period  $n$  of a periodic sequence was called a "proper decimation" on p.78 in his book. For example,  $(s_1, s_2, s_3, s_4, s_5) = (0, 0, 0, 1, 1)$  is a sequence with length 5. Now 2 is relatively prime to 5 and a 2-step shift generates the new sequence  $(s_2, s_4, s_1, s_3, s_5) = (0, 1, 0, 0, 1)$ .

$D_n = \{Q_{c,1}, Q_{c,-1} \mid c \in \{0, 1, \dots, n-1\}\}$ . This is the group of all possible combinations of cyclic shifts and reversals of sequences.  $D_n$  is known as the dihedral group.

$E_n = \{Q_{c,k} \mid 1 \leq k < n, (n, k) = 1, c \in \{0, 1, \dots, n-1\}\}$ . I call these transformations step cyclic shifts. These are the most general class of linear transformations which are permutations of a sequence. They incorporate reversals and the dihedral group because  $n-1$  is relatively prime to  $n$  and so  $Q_{c,-1}$  has the same effect as  $Q_{c,n-1}$ . Step cyclic shifts are used to define equivalence amongst cyclic difference sets, e.g. Baumert (1971, p.2).

The group property of the above sets under the operation of function composition has been indicated by previous authors, e.g. stated by Gilbert and Riordan (1961) for  $C_n$  and  $D_n$ , implied by Titsworth (1964) for  $H_n$ , and stated by Titsworth (1964) for  $E_n$ . The group properties are readily established by using the formula in Lemma 2.1 above. For example, for  $E_n$  we have:

*Closure:* Consider  $Q_{c_2, k_2} Q_{c_1, k_1} = Q_{c_1+k_1 c_2, k_1 k_2}$  where  $Q_{c_2, k_2}, Q_{c_1, k_1} \in E_n$ . We note that  $(k_1, n) = 1, (k_2, n) = 1$  together imply that  $(k_1 k_2, n) = 1$ , and  $c_1 + k_1 c_2$  is equivalent modulo  $n$  to a number in the range  $0, 1, \dots, n-1$ . Therefore  $Q_{c_1+k_1 c_2, k_1 k_2} \in E_n$ .

$$\text{Associativity: } \left\{ \begin{array}{l} Q_{c_3, k_3}(Q_{c_2, k_2} Q_{c_1, k_1}) = Q_{c_3, k_3} Q_{c_1+k_1 c_2, k_1 k_2} \\ \quad = Q_{c_1+k_1 c_2+k_1 k_2 c_3, k_1 k_2 k_3} \\ (Q_{c_3, k_3} Q_{c_2, k_2})(Q_{c_1, k_1}) = Q_{c_2+k_2 c_3, k_2 k_3} Q_{c_1, k_1} \\ \quad = Q_{c_1+k_1 c_2+k_1 k_2 c_3, k_1 k_2 k_3} \end{array} \right.$$

*Inverse:* From the Euclidean algorithm, e.g. Niven and Zuckerman (1972, p.7), if  $(k_1, n) = 1$  then there exists a  $k_2$  such that  $(k_2, n) = 1$  and  $k_1 k_2 \equiv 1 \pmod{n}$ . Now for arbitrary  $Q_{c_1, k_1}$  its inverse will be  $Q_{-k_2 c_1, k_2}$  since  $Q_{-k_2 c_1, k_2} Q_{c_1, k_1} = Q_{c_1-k_1 k_2 c_1, k_1 k_2} = Q_{c_1-c_1, 1} = Q_{0,1}$ .

With regard to permutations of the alphabet, for the moment we merely note that if a permutation of  $A_q$  changes the character  $s_i$  to character  $t_i$  then the sequence  $(s_1, s_2, \dots, s_n)$  is changed to the sequence  $(t_1, t_2, \dots, t_n)$ .

Two sequences  $S_1, S_2$  are said to be dihedrally equivalent if one can be obtained from the other by some combination of cyclic shifts and reversals, i.e. by the action of  $D_n$ . We shall write  $S_1 \equiv_D S_2$ . Two sequences are said to be necklace equivalent if one can be obtained

from the other by some combination of permuting the alphabet, cyclic shifts or reversals. In this case we shall write  $S_1 \equiv_N S_2$ .

## 2.3 Step cyclic shifts and dihedral equivalence

The following lemma was first published in Nester (1997) and will be required in Chapter 3. The lemma proves that arbitrary step cyclic shifts preserve dihedral equivalence.

**Lemma 2.2.** If  $S_1 \equiv_D S_2$  then  $Q_{c,k}(S_1) \equiv_D Q_{c,k}(S_2)$  for all integers  $c, k$ , where  $k$  is relatively prime to the lengths (or periods) of the sequences.

*Proof:* It is sufficient to separately consider the cases  $S_1 = Q_{f,1}(S_2)$  and  $S_1 = Q_{0,-1}(S_2)$ .

Case (1): Suppose  $S_1 = Q_{f,1}(S_2)$ , i.e.  $S_1$  is an arbitrary cyclic shift of  $S_2$ . From Lemma 2.1,  $Q_{c,k}Q_{f,1} = Q_{f+c,k}$ . Also,  $Q_{g,1}Q_{c,k} = Q_{c+kg,k} = Q_{f+c,k}$  provided we choose  $g$  such that  $kg \equiv f \pmod{n}$ . This is possible since  $k$  is relatively prime to  $n$ .

Thus  $Q_{c,k}(S_1) = Q_{g,1}Q_{c,k}(S_2)$  and therefore  $Q_{c,k}(S_1) \equiv_D Q_{c,k}(S_2)$ .

Case (2): Suppose  $S_1 = Q_{0,-1}(S_2)$ , i.e.  $S_1$  is a reversal of  $S_2$ . From Lemma 2.1,  $Q_{c,k}Q_{0,-1} = Q_{-c,-k}$ . Also,  $Q_{g,1}Q_{0,-1}Q_{c,k} = Q_{g,1}Q_{c,-k} = Q_{c-k,g,-k} = Q_{-c,-k}$  provided we choose  $g$  such that  $kg \equiv 2c \pmod{n}$ . Thus  $Q_{c,k}(S_1) = Q_{g,1}Q_{0,-1}Q_{c,k}(S_2)$  and therefore  $Q_{c,k}(S_1) \equiv_D Q_{c,k}(S_2)$ .

The following lemma, also from Nester (1997), shows that applying a  $k$ -step cyclic shift to a sequence is essentially the same as applying an  $(n - k)$ -step cyclic shift.

**Lemma 2.3.** Consider any sequence  $S$  of length, or period,  $n$ . For arbitrary integers  $c, d$  and arbitrary  $k$  relatively prime to  $n$  we have  $Q_{c,k}(S) \equiv_D Q_{d,n-k}(S)$ .

*Proof:*  $Q_{f,1}Q_{0,-1}Q_{d,n-k} = Q_{f,1}Q_{d,-n+k} = Q_{f,1}Q_{d,k} = Q_{d+kf,k} = Q_{c,k}$  provided  $f$  is chosen so that  $d + kf \equiv c \pmod{n}$ .

It is also trivially easy to show that for arbitrary  $k$  relatively prime to  $n$  and for arbitrary  $c, d$ ,  $S$ , we have  $Q_{c,k}(S) \equiv_D Q_{d,k}(S)$ . Thus the dihedral equivalence class of  $Q_{c,k}(S)$  is determined by the step  $k$  and is not influenced by the shift  $c$ . Combining this observation with Lemma 2.1, for arbitrary  $c, d, e$  we have  $Q_{c,k_2} Q_{d,k_1}(S) \equiv_D Q_{e,k_1 k_2}(S)$ . This sets up a natural homomorphism between the group of step cyclic shifts when acting on dihedral equivalence classes of sequences, and the group  $G$  of integers relatively prime to  $n$  under the operation of multiplication modulo  $n$ . Using standard group theory results, it follows that for a fixed sequence  $S$ ,  $\{k \mid Q_{c,k}(S) \equiv_D S\}$  is a subgroup of  $G$ . Furthermore, if we denote this subgroup by  $G_S$  then  $Q_{c,k_1}(S) \equiv_D Q_{c,k_2}(S)$  provided  $k_1$  and  $k_2$  belong to the same coset of  $G_S$ .

If  $k$  is relatively prime to  $n$  and if  $Q_{c,k}(S) \equiv_D S$  then we shall say that  $k$  stabilizes  $S$ . Since all such  $k$  form a subgroup we can assert that  $k$  stabilizes  $S$  if and only if  $k^{-1}$  stabilizes  $S$ . Furthermore, a consequence of Lemma 2.3 is that  $k$  stabilizes  $S$  if and only if  $n - k$  stabilizes  $S$ .

**Example 2.3:** Suppose  $S = (0, 1, 0, 0, 1)$ . We have

$$Q_{0,1}(S) = S = (0, 1, 0, 0, 1),$$

$$Q_{0,2}(S) = (s_{0+1 \times 2}, s_{0+2 \times 2}, s_{0+3 \times 2}, s_{0+4 \times 2}, s_{0+5 \times 2}) = (s_2, s_4, s_1, s_3, s_5) = (1, 0, 0, 0, 1),$$

$$Q_{0,3}(S) = (s_{0+1 \times 3}, s_{0+2 \times 3}, s_{0+3 \times 3}, s_{0+4 \times 3}, s_{0+5 \times 3}) = (s_3, s_1, s_4, s_2, s_5) = (0, 0, 0, 1, 1), \text{ and}$$

$$Q_{0,4}(S) = (s_{0+1 \times 4}, s_{0+2 \times 4}, s_{0+3 \times 4}, s_{0+4 \times 4}, s_{0+5 \times 4}) = (s_4, s_3, s_2, s_1, s_5) = (0, 0, 1, 0, 1).$$

Notice that  $Q_{0,1}(S)$  and  $Q_{0,4}(S)$  are dihedrally equivalent to  $S$ . Also  $Q_{0,2}(S)$  and  $Q_{0,3}(S)$  are dihedrally equivalent to each other but not equivalent to  $S$ . Thus 1 and 4 are the only stabilizers of  $S$ . We also note that:

- $\{1, 4\}$  form a subgroup of the multiplicative group  $\{1, 2, 3, 4\}$  modulo 5;
- 4 must be a stabilizer in any case because 1 is always a stabilizer and  $4 = 5 - 1$ ;
- $4^{-1} = 4$  which confirms that  $4^{-1}$  is a stabilizer;
- 2 is not a stabilizer and this implies that 3 is not a stabilizer because  $3 = 5 - 2$ ; and
- 2 is not a stabilizer and this implies that 3 is not a stabilizer because  $2^{-1} = 3$ .

## 2.4 Polya counting methods

Consider a group  $G$  of permutations acting on a finite set  $X$ . This means that for arbitrary  $\pi_1, \pi_2 \in G$  and arbitrary  $x \in X$  we have  $\pi_1 x \in X$  and  $\pi_1(\pi_2 x) = (\pi_1 \pi_2)x$  where  $\pi_1 \pi_2$  denotes the usual composition of permutations. It is well-known that all permutations can be decomposed into disjoint cycles. Let  $n$  denote  $|X|$ , the order of  $X$ . If a permutation  $\pi$  is decomposed into  $j_1$  cycles of length 1,  $j_2$  cycles of length 2,  $\dots$ ,  $j_n$  cycles of length  $n$ , then  $\pi$  is said to have cycle type  $(j_1, j_2, \dots, j_n)$ . Furthermore, the monomial

$$P(\pi) = x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} = \prod_{i=1}^n x_i^{j_i}$$

is said to be the cycle polynomial, or cycle structure representation, of the permutation  $\pi$ .

Note that in some applications, such as de Bruijn's theorem discussed below, it is more convenient to use the infinite form of the cycle type, viz.

$$(j_1, j_2, \dots, j_n, j_{n+1}, j_{n+2}, \dots) = (j_1, j_2, \dots, j_n, 0, 0, \dots).$$

**Example 2.4:** Continuing examples 2.1 and 2.2, we suppose that the permutation

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix}$  is a permutation of five sequence positions. It can be decomposed into the

disjoint cycles  $(1\ 4\ 5\ 2)$  and  $(3)$ . The first cycle is interpreted as 1 is replaced by 4, 4 is replaced by 5, 5 is replaced by 2 and 2 is replaced by 1. The second cycle means that 3 is left unchanged. This permutation has one cycle of length 1 and 1 cycle of length 4. Its cycle type is therefore  $(1, 0, 0, 1, 0)$ , or the infinite form  $(1, 0, 0, 1, 0, 0, 0, 0, \dots)$ . Its cycle polynomial

is  $x_1^1 x_4^1 = x_1 x_4$ .

The cycle index of the group  $G$  of permutations is simply defined to be the arithmetic average of all of the cycle polynomials, i.e.  $P_G(x_1, x_2, \dots, x_n) = \frac{1}{|G|} \sum_{\pi \in G} P(\pi)$ .

Now consider the set  $R^X$  of all functions which map  $X$  into the finite set  $R$ . In our particular situation,  $X$  will be the set of  $n$  positions in a sequence,  $R$  will be a set of  $q$  colourings, or equivalently an alphabet with  $q$  characters, and  $R^X$  can be regarded as the set of all possible sequences of length  $n$  using the given alphabet.  $R$  will have exactly two elements if we are considering binary sequences,  $R$  will have exactly three elements if we are considering ternary sequences, and so on.

The permutations in  $G$  induce equivalence classes in  $R^X$ . Each equivalence class is frequently called a pattern. If  $c_1, c_2, \dots, c_q$  are variables which denote the elements of  $R$  then Polya's enumeration theorem states that the generating function for the so-called pattern

inventory is  $W = P_G \left( \sum_{i=1}^q c_i, \sum_{i=1}^q c_i^2, \dots, \sum_{i=1}^q c_i^n \right)$ , e.g. Street and Wallis (1977, p.586).

Using this generating function, the number of patterns, or equivalence classes, in which  $l_1$  elements of  $X$  are "coloured"  $c_1$ ,  $l_2$  elements of  $X$  are "coloured"  $c_2$ , and so on, is given by the coefficient of  $c_1^{l_1} c_2^{l_2} \dots c_n^{l_n}$  in  $W$ . The total number of patterns, or equivalence classes, induced by  $G$  can be found by substituting 1 for all  $c_i$  in the pattern inventory. Therefore the total number of equivalence classes is  $W = P_G(q, q, \dots, q)$ .

We shall also need a generalization of Polya's enumeration theorem due to de Bruijn (1959, not sighted) and which is discussed in the review article by de Bruijn (1964). In this generalization we assume that there is also a permutation group  $H$  which is acting on the set  $R$  of colours. If permutation  $h$  of  $H$  has the infinite cycle type  $(j_{h_1}, j_{h_2}, \dots, j_{h_q}, 0, 0, \dots)$  then the "constant form" of de Bruijn's theorem states that the number of patterns, or equivalence classes, induced by the combined effects of  $G$  on  $X$  and  $H$  on  $R$  is given by

$$\frac{1}{|H|} \sum_{h \in H} P_G \left( \sum_{k|1} k j_{h_k}, \sum_{k|2} k j_{h_k}, \dots, \sum_{k|n} k j_{h_k} \right).$$

In this thesis we are only interested in all possible permutations of the colours  $R$ , i.e.  $H$  will be the symmetric group  $S_q$ , and we will confine ourselves to the range from the trivial case

$q = 1$  through to  $q = 6$ .

For example, suppose we are considering sequences of length  $n = 8$  based on  $q = 6$  colours.

A particular permutation  $h$  of  $H$  must have a cycle type of the form  $(j_1, j_2, \dots, j_6)$ . Now when we evaluate the constant form of de Bruijn's theorem we use

$$\sum_{k|1} k j_k = j_1,$$

$$\sum_{k|2} k j_k = j_1 + 2 j_2,$$

$$\sum_{k|3} k j_k = j_1 + 3 j_3,$$

$$\sum_{k|4} k j_k = j_1 + 2 j_2 + 4 j_4,$$

$$\sum_{k|5} k j_k = j_1 + 5 j_5,$$

$$\sum_{k|6} k j_k = j_1 + 2 j_2 + 3 j_3 + 6 j_6,$$

$$\sum_{k|7} k j_k = j_1 + 7 j_7 = j_1, \text{ because restricted to } q \leq 6,$$

$$\sum_{k|8} k j_k = j_1 + 2 j_2 + 4 j_4 + 8 j_8 = j_1 + 2 j_2 + 4 j_4.$$

Now a collection of positive integers  $t_i$  which add up to  $q$  is called a partition of  $q$ . Thus suppose  $t_1 + t_2 + \dots + t_f = q$ . Then the partition of  $n$  is said to have type  $(j_1, j_2, \dots, j_q)$  if exactly  $j_1$  of the  $t_i$ 's are equal to 1, exactly  $j_2$  of the  $t_i$ 's are equal to 2, and so on. It is well-known, e.g. Riordan (1958, p.67), that there is a one-to-one correspondence between the cycle types of permutations in  $S_q$  and the types of the partitions of  $q$ . Furthermore, the number of permutations in  $S_q$  with cycle type  $(j_1, j_2, \dots, j_q)$  is given by

$\frac{q!}{j_1! j_2! \dots j_q! 1^{j_1} 2^{j_2} \dots q^{j_q}}$ . This information gleaned from the partitions of  $q$  is summarized in

Table 2.1 and was incorporated in the Maple V scripts which I used to evaluate the constant form of the power enumeration theorem. Maple V (1981-1993) is a computer program for symbolic manipulation and numerical calculations.

*Table 2.1. Cycle decompositions for the symmetric groups of orders 1 through to 6.*

$q$	Partition	Cycle type	Frequency of cycle type
1	1	(1,0,0,0,0,0)	1
2	1+1	(2,0,0,0,0,0)	1
	2	(0,1,0,0,0,0)	1
3	1+1+1	(3,0,0,0,0,0)	1
	2+1	(1,1,0,0,0,0)	3
	3	(0,0,1,0,0,0)	2
4	1+1+1+1	(4,0,0,0,0,0)	1
	2+1+1	(2,1,0,0,0,0)	6
	2+2	(0,2,0,0,0,0)	3
	3+1	(1,0,1,0,0,0)	8
	4	(0,0,0,1,0,0)	6
5	1+1+1+1+1	(5,0,0,0,0,0)	1
	2+1+1+1	(3,1,0,0,0,0)	10
	2+2+1	(1,2,0,0,0,0)	15
	3+1+1	(2,0,1,0,0,0)	20
	3+2	(0,1,1,0,0,0)	20
	4+1	(1,0,0,1,0,0)	30
	5	(0,0,0,0,1,0)	24
6	1+1+1+1+1+1	(6,0,0,0,0,0)	1
	2+1+1+1+1	(4,1,0,0,0,0)	15
	2+2+1+1	(2,2,0,0,0,0)	45
	2+2+2	(0,3,0,0,0,0)	15
	3+1+1+1	(3,0,1,0,0,0)	40
	3+2+1	(1,1,1,0,0,0)	120
	3+3	(0,0,2,0,0,0)	40
	4+1+1	(2,0,0,1,0,0)	90
	4+2	(0,1,0,1,0,0)	90
	5+1	(1,0,0,0,1,0)	144
	6	(0,0,0,0,0,1)	120

## 2.5 Cycle polynomials of permutations based on linear transformations

For the rest of this chapter we will be concerned with counting equivalence classes of sequences. Thus the finite set  $X$  of the previous section will be the set of positions in a sequence of length  $n$ , or the set of positions in the finite sequence representation of a periodic sequence of period  $n$ . Furthermore, the group  $G$  acting on  $X$  will be one of the groups of linear transformations introduced earlier.

Of special note below is the fact that when computing greatest common divisors we assume that  $(j, 0) = j$  for all  $j > 0$ .

The identity permutation of degree  $n$  leaves all elements fixed so there are  $n$  cycles of length 1. Thus the cycle polynomial of the identity permutation is  $x_1^n$ .

For the general case  $Q_{c,k}$  where  $(k, n) = 1$  we base our results on those of Titsworth (1964).

If  $(k, d) = 1$  then we use  $E(k, d)$  to denote the exponent of  $k$  modulo  $d$ , i.e.  $E(k, d)$  is the minimum  $\delta$ ,  $\delta \geq 1$ , such that  $k^\delta \equiv 1 \pmod{d}$ . We note that  $E(1, d) = E(k, 1) = 1$ . Also  $(d-1)^2 = d^2 - 2d + 1 \equiv 1 \pmod{d}$  so  $E(d-1, d) = 2$  for  $d > 1$ .

We also let  $P(k, d)$  denote the least positive integer  $\delta$  such that

$k^{\delta-1} + k^{\delta-2} + \dots + k^2 + k^1 + 1 \equiv 0 \pmod{d}$ . I call  $P(k, d)$  the proponent of  $k$  modulo  $d$ . We note that  $P(1, d) = d$  because  $P(1, d)$  is the minimum  $\delta$  such that  $1^{\delta-1} + 1^{\delta-2} + \dots + 1^2 + 1^1 + 1 \equiv 0 \pmod{d}$ , i.e. the minimum  $\delta$  such that  $\delta \times 1 = d$ . We also note that  $P(k, 1) = 1$  and that  $P(d-1, d) = 2$  for  $d > 1$ .

Later on the following lemma will be required.

**Lemma 2.4.**  $E(k, n) = P\left(k, \frac{n}{(n, k-1)}\right)$ . Equivalently,  $P(k, n) = E(k, n(n, k-1))$ . *Proof:*

$E(k, n)$  is the minimum  $\delta$  such that  $k^\delta \equiv 1 \pmod{n}$ . Therefore  $k^\delta - 1 \equiv 0 \pmod{n}$  and factorizing gives  $(k-1)(k^{\delta-1} + \dots + k + 1) \equiv 0 \pmod{n}$ .

Since  $(n, k-1) \mid (k-1)$  we must have  $\frac{n}{(n, k-1)} \mid (k^{\delta-1} + \dots + k + 1)$ .

Therefore  $\delta \geq P\left(k, \frac{n}{(n, k-1)}\right)$ ,

$$\text{i.e. } E(k, n) \geq P\left(k, \frac{n}{(n, k-1)}\right). \quad (2.1)$$

Also,  $P\left(k, \frac{n}{(n, k-1)}\right)$  is the minimum  $\delta'$  such that  $k^{\delta'-1} + \dots + k^1 + 1 \equiv 0 \pmod{\frac{n}{(n, k-1)}}$ .

Obviously  $(n, k-1) \mid (k-1)$  and so  $\frac{n}{(n, k-1)}(n, k-1) \mid (k^{\delta'-1} + \dots + k^1 + 1)(k-1)$ .

Therefore  $n \mid (k^{\delta'} - 1)$ , i.e.  $\delta' \geq E(k, n)$ ,

$$\text{i.e. } P\left(k, \frac{n}{(n, k-1)}\right) \geq E(k, n). \quad (2.2)$$

Now equations (2.1) and (2.2) together prove the result.

Titsworth has shown that  $P(k, d) = \frac{d E(k, d)}{(d, k^{E(k, d)-1} + \dots + k + 1)}$ . Titsworth also proved the

following important lemma.

**Lemma 2.5.** Consider the permutation corresponding to an arbitrary linear transformation

$Q_{c,k}$  where  $(k, n) = 1$ . The cycle containing position  $i$  has length  $P\left(k, \frac{n}{(n, (k-1)i+c)}\right)$ .

From this lemma we can immediately deduce the following corollary which is also due to Titsworth.

**Corollary 2.6.** The cycle polynomial for an arbitrary linear transformation  $Q_{c,k}$  with

$$(k, n) = 1 \text{ is } \prod_{i=1}^n x_{m(i)}^{1/m(i)} \text{ where } m(i) = P\left(k, \frac{n}{(n, (k-1)i+c)}\right).$$

**Example 2.5:** Continuing examples 2.1 and 2.4, the linear transformation  $Q_{2,2}(S)$  can be

expressed as the permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix}$ . This permutation has disjoint cycles

$(1 4 5 2)$  and  $(3)$ . The cycle containing 4, say, has length 4. According to Lemma 2.5, the

cycle containing 4 should have length  $P\left(k, \frac{n}{(n, (k-1)i+c)}\right) = P\left(2, \frac{5}{(5, (2-1)4+2)}\right) = P(2,5)$ .

From the definition of the proponent,  $P(2,5)$  is the least positive integer  $\delta$  such that  $2^{\delta-1} + 2^{\delta-2} + \dots + 2^2 + 2^1 + 1 \equiv 0 \pmod{5}$ . This can be solved by trial and error to yield  $P(2,5) = 4$ , as it should. Now with regard to Corollary 2.6, for positions 1, 2, 3, 4, 5 the  $m(i)$  are 4, 4, 1, 4, and 4 respectively. Therefore the cycle polynomial for  $Q_{2,2}(S)$  should be  $x_4^{1/4}x_4^{1/4}x_1^{1/1}x_4^{1/4}x_4^{1/4} = x_1x_4$ , and this agrees with the answer in example 2.4.

From Corollary 2.6 we immediately have the following theorem which is implicit in Titsworth.

**Theorem 2.7.** If  $A$  is a group of linear transformations of the form  $Q_{c,k}$  with  $(k,n)=1$ , then

the cycle index of  $A$  is  $P_A(x_1, \dots, x_n) = \frac{1}{|A|} \sum_{Q_{c,k} \in A} \prod_{i=1}^n x_{m(i)}^{1/m(i)}$  where

$$m(i) = P\left(k, \frac{n}{(n, (k-1)i+c)}\right).$$

## 2.6 Cycle indices for the groups of linear transformations

### 2.6.1 Cycle index for I

This is trivial but we demonstrate the use of Theorem 2.7.

**Theorem 2.8.** The cycle index for  $I$ , the group consisting solely of the identity permutation, is  $x_1^n$ .

*Proof:* Group  $I$  consists of the sole linear transformation  $Q_{0,1}$ . The corresponding proponent

$$\text{is } P\left(k, \frac{n}{(n, (k-1)i+c)}\right) = P\left(1, \frac{n}{(n, 0i+0)}\right) = P\left(1, \frac{n}{n}\right) = 1.$$

Therefore  $P_I(x_1, \dots, x_n) = \frac{1}{n} \prod_{i=1}^n x_i^{1/1} = x_1^n$ .

## 2.6.2 Cycle index for $C_n$

This cycle index is well-known, e.g. Polya and Read (1987, p.22), but we shall again demonstrate use of Theorem 2.7. In this and some later sections we shall refer to Euler's totient function, e.g. Niven and Zuckerman (1972, p.22), for which  $\varphi(d)$  denotes the number of positive integers less than  $d$  which are relatively prime to  $d$ .

**Theorem 2.9.** The cycle index for  $C_n$  is  $\frac{1}{n} \sum_{d|n} \varphi(d) x_d^{n/d}$ .

*Proof:* All linear transformations in  $C_n$  are of the form  $Q_{c,1}$  where  $0 \leq c < n$ . Now

$$(n, (k-1)i + c) = (n, 0i + c) = (n, c).$$

If  $(n, c) = d$  then  $\left(\frac{n}{d}, \frac{c}{d}\right) = 1$  and there are  $\varphi\left(\frac{n}{d}\right)$  different such possible values for  $c$ .

We have  $P\left(k, \frac{n}{(n, (k-1)i + c)}\right) = P\left(1, \frac{n}{(n, c)}\right) = P\left(1, \frac{n}{d}\right) = \frac{n}{d}$ .

$$\begin{aligned} \text{Therefore } P_{C_n}(x_1, \dots, x_n) &= \frac{1}{n} \sum_{c=0}^{n-1} \prod_{i=1}^n x_{m(i)}^{1/m(i)} \\ &= \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) \prod_{i=1}^n x_{m(i)}^{1/m(i)} \\ &= \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) \prod_{i=1}^n x_{n/d}^{1/(n/d)} \\ &= \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) x_{n/d}^{n/(n/d)} \\ &= \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) x_{n/d}^d \\ &= \frac{1}{n} \sum_{d|n} \varphi(d) x_d^{n/d}, \end{aligned}$$

where the last step is obtained by swapping  $d$  and  $n/d$ .

### 2.6.3 Cycle index for $R_n$

This cycle index is trivially easy to generate by inspection of the permutations. However, we shall use the general formula provided by Theorem 2.7, thereby lending credence to the maxim that general methods always work but direct methods are often simpler!

**Theorem 2.10.** The cycle index for  $R_n$  is

$$P_{R_n}(x_1, \dots, x_n) = \frac{x_1^n + x_2^{n/2}}{2} \text{ for } n \in E \text{ where } E \text{ denotes the set of positive even integers; and}$$

$$P_{R_n}(x_1, \dots, x_n) = \frac{x_1^n + x_1 x_2^{(n-1)/2}}{2} \text{ for } n \in O \text{ where } O \text{ denotes the set of positive odd integers.}$$

*Proof:*  $R_n = \{Q_{0,1}, Q_{1,-1}\}$ . We have already shown that the cycle polynomial for  $Q_{0,1}$  is

$x_1^n$ . For  $Q_{1,-1}$  we have the proponent

$$P\left(k, \frac{n}{(n, (k-1)i+c)}\right) = P\left(n-1, \frac{n}{(n, -2i+1)}\right) = P\left(n-1, \frac{n}{(n, 2i-1)}\right).$$

Now if  $n \in O$  and if  $i = \frac{n+1}{2}$  then the proponent reduces to

$$P\left(n-1, \frac{n}{(n, 2i-1)}\right) = P\left(n-1, \frac{n}{(n, n)}\right) = P\left(n-1, \frac{n}{n}\right) = P(n-1, 1) = 1.$$

For  $n \in O$  and all other relevant values of  $i$ , or for  $n \in E$  and all relevant values of  $i$ , we must

have  $2i-1 < 2n$  and  $2i-1 \neq n$  so that  $(n, 2i-1) < n$  and therefore  $\frac{n}{(n, 2i-1)} > 1$ .

If  $f$  is an arbitrary divisor of  $n$  then we have the trivial observation that  $n \equiv \frac{n}{f} \pmod{\frac{n}{f}}$  and

this proves that  $n-1 \equiv \frac{n}{(n, 2i-1)} - 1 \pmod{\frac{n}{(n, 2i-1)}}$ . Also, from the definition of a

proponent, if  $k_1 \equiv k_2 \pmod{d}$  then  $P(k_1, d) = P(k_2, d)$ . Thus for these remaining values of

$i$  and  $n$  our proponent  $P\left(n-1, \frac{n}{(n, 2i-1)}\right)$  reduces  $P\left(\frac{n}{(n, 2i-1)} - 1, \frac{n}{(n, 2i-1)}\right) = 2$ .

Therefore from Corollary 2.6, for  $n \in O$  the cycle polynomial for  $Q_{1,-1}$  must be

$x_1^{1/2} \left(x_2^{1/2}\right)^{n-1} = x_1 x_2^{(n-1)/2}$ , and for  $n \in E$  the cycle polynomial for  $Q_{1,-1}$  must be

$$\left(x_2^{1/2}\right)^n = x_2^{n/2}.$$

The theorem now follows by taking the average of the cycle polynomials for the two linear transformations.

#### 2.6.4 Cycle index for $D_n$

The cycle index for the dihedral group is also well-known, e.g. Polya and Read (1987, p.22), and we merely quote it.

**Theorem 2.11.** The cycle index for  $D_n$  is

$$P_{D_n}(x_1, \dots, x_n) = \frac{1}{2n} \sum_{d|n} \varphi(d) x_d^{n/d} + \frac{x_1 x_2^{(n-1)/2}}{2} \text{ for } n \in O, \text{ and}$$

$$P_{D_n}(x_1, \dots, x_n) = \frac{1}{2n} \sum_{d|n} \varphi(d) x_d^{n/d} + \frac{x_1^2 x_2^{(n-2)/2}}{4} + \frac{x_2^{n/2}}{4} \text{ for } n \in E.$$

#### 2.6.5 Cycle index for $H_n$

The cycle index for step shifts is implicit in Titsworth. Titsworth did not use Theorem 2.7 but resorted to first principles by counting cycles of transformations directly.

**Theorem 2.12.** The cycle index for  $H_n$  is

$$P_{H_n}(x_1, \dots, x_n) = \frac{1}{\varphi(n)} \sum_{(k,n)=1} \prod_{d|n} x_{E(k,d)}^{\varphi(d)/E(k,d)} \text{ where } 1 \leq k < n.$$

#### 2.6.6 Cycle index for $E_n$

It appears that Theorem 2.7 can be simplified only slightly when calculating the cycle index for the general case of all step cyclic shifts,  $E_n$ . The following theorem is not in Titsworth.

**Theorem 2.13.** The cycle index for  $E_n$  is

$$P_{E_n}(x_1, \dots, x_n) = \frac{1}{n\varphi(n)} \sum_{(k,n)=1} \sum_{c=1}^{(n,k-1)} \frac{n}{(n, k-1)} \prod_{i=1}^n x_{m(i)}^{1/m(i)} \text{ where } 1 \leq k < n \text{ and}$$

$$m(i) = P\left(k, \frac{n}{(n, (k-1)i + c)}\right).$$

*Proof:* From Theorem 2.7,  $P_{E_n}(x_1, \dots, x_n) = \frac{1}{|E_n|} \sum_{Q_{c,k} \in E_n} \prod_{i=1}^n x_{m(i)}^{1/m(i)}$ . Since

$$E_n = \left\{ Q_{c,k} \mid 1 \leq k < n, (n, k) = 1, 0 \leq c < n \right\} \text{ and since } |E_n| = n\varphi(n) \text{ we immediately have}$$

$$P_{E_n}(x_1, \dots, x_n) = \frac{1}{n\varphi(n)} \sum_{(k,n)=1} \sum_{c=0}^{n-1} \prod_{i=1}^n x_{m(i)}^{1/m(i)}. \quad (2.3)$$

We now consider the possible values of  $(k-1)i + c$  used in the proponent. For fixed  $k$  the values of  $(k-1)i$  are  $(k-1), (k-1)2, \dots$ , and eventually  $(k-1)i \equiv 0 \pmod{n}$ . This occurs

when  $i = \frac{n}{(n, k-1)}$ . Thereafter the values of  $(k-1)i$  repeat modulo  $n$ , i.e. we have modulo  $n$ :

$$(k-1), (k-1)2, \dots, (k-1)\frac{n}{(n, k-1)}, (k-1), (k-1)2, \dots.$$

Now considering the effects modulo  $n$  on  $(k-1)i + c$  of different possible values of  $c$  we have:

$$c=0: \quad (k-1), (k-1)2, \dots, (k-1)\frac{n}{(n, k-1)}, (k-1), \dots;$$

$$c=1: \quad (k-1)+1, (k-1)2+1, \dots, (k-1)\frac{n}{(n, k-1)}+1, (k-1)+1, \dots; \text{ and so on.}$$

Thus for each value of  $c$  we have a set of  $\frac{n}{(n, k-1)}$  different elements, viz.

$T_c = \left\{ (k-1)i + c \mid 1 \leq i \leq \frac{n}{(n, k-1)} \right\}$  where we assume that all elements of  $T_c$  are reduced modulo  $n$ .

Now either  $T_{c_1} = T_{c_2}$  or  $T_{c_1} \cap T_{c_2} = \emptyset$  where  $\emptyset$  denotes the empty set. For suppose

$T_{c_1} \cap T_{c_2} \neq \emptyset$  then there exists at least one pair of values  $i$  such that

$$(k-1)i_1 + c_1 = (k-1)i_2 + c_2 \text{ where } (k-1)i_1 + c_1 \in T_{c_1} \text{ and } (k-1)i_2 + c_2 \in T_{c_2}. \text{ Let}$$

$(k-1)i_1' + c_1$  be any other element of  $T_{c_1}$  then

$$\begin{aligned}(k-1)i_1' + c_1 &= (k-1)(i_1' - i_1 + i_1) + c_1 \\&= (k-1)(i_1' - i_1) + (k-1)i_1 + c_1 \\&= (k-1)(i_1' - i_1) + (k-1)i_2 + c_2 \\&= (k-1)(i_1' - i_1 + i_2) + c_2.\end{aligned}$$

Therefore  $T_{c_1} \subset T_{c_2}$ . Similarly,  $T_{c_2} \subset T_{c_1}$  and so the  $T_c$ 's are either disjoint or identical.

Now suppose that  $(k-1)i_1 + c_1 \equiv (k-1)i_2 + c_2 \pmod{n}$  where  $c_1 \neq c_2$ .

Then  $(k-1)(i_1 - i_2) \equiv c_2 - c_1 \pmod{n}$ . (2.4)

Now  $i_1 - i_2 = 0$  would imply that  $c_2 - c_1 \equiv 0 \pmod{n}$  which is impossible because

$0 \leq c_1, c_2 < n$ . Therefore  $i_1 - i_2 \neq 0$  and using this fact in equation (2.4) we conclude that

$$(n, k-1) \mid (c_2 - c_1).$$

Thus the sets  $T_0, T_1, \dots, T_{(n, k-1)-1}$  are all distinct and then the sets start repeating for higher

values of  $c$ , i.e.  $T_0 = T_{(n, k-1)}$  etc. and each set is repeated  $\frac{n}{(n, k-1)}$  times.

Thus the symbol  $\sum_{c=0}^{n-1}$  in equation (2.3) reduces to  $\sum_{c=0}^{(n, k-1)-1} \frac{n}{(n, k-1)} = \sum_{c=1}^{(n, k-1)} \frac{n}{(n, k-1)}$  and

the theorem is proved.

If  $n$  is prime then the cycle index for  $E_n$  has a relatively simple form which is implicit in Titsworth.

**Corollary 2.14.** If  $p$  is prime then

$$P_{E_p}(x_1, \dots, x_n) = \frac{1}{p(p-1)} \left\{ x_1^p + (p-1)x_p + p \sum_{\substack{d \mid (p-1) \\ d > 1}} \varphi(d) x_1 x_d^{(p-1)/d} \right\}.$$

## 2.7 Counting primitive sequences

Suppose we are enumerating sequences with a particular property. Let  $F(n)$  be the total number of finite sequences of length  $n$  with this property, or the total number of periodic sequences of period  $n$  with this property. Let  $f(d)$  denote the corresponding number of primitive sequences with this same property. Then obviously  $F(n) = \sum_{d|n} f(d)$ . Thus we can

use the well-known and remarkable Möbius inversion formula  $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$ , e.g.

Niven and Zuckerman (1972, p.88), to enumerate primitive sequences, where the Möbius function  $\mu$  takes on the following values:

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{if } p^2 \mid d \text{ for some prime } p, \\ (-1)^k & \text{if } d \text{ is the product of } k \text{ distinct primes.} \end{cases}$$

Note that we do not need to evaluate the cycle indices for the  $F\left(\frac{n}{d}\right)$ 's before substituting into the Möbius inversion formula so we can, in effect, generate "cycle indices" for the primitive sequences, i.e. we can express both  $F\left(\frac{n}{d}\right)$  and  $f(n)$  as polynomials in the  $x_i$ 's.

## 2.8 Counting sequences which have exactly $q$ different characters

Consider sequences of length or period  $n$  based on an alphabet which has exactly  $q$  different characters, where  $q \leq n$ . For  $i \leq q$  let  $\#(\leq i)$  denote the total number of sequences with a desired property and which contain  $i$  or fewer different characters. Also let  $\#i$  denote the total number of sequences with a desired property and which contain exactly  $i$  different characters.

Initially suppose we are in a situation where we are not considering sequence structures, i.e. we are in a situation where any new permutation of the alphabetical characters will generate an essentially new sequence. With regard to the Polya and de Bruijn counting methods we

are in a situation where  $G \times H = G \times I_q$ .

Now the  $i$  different letters can be chosen from the alphabet in  $\binom{q}{i}$  ways and we immediately have  $\#(\leq q) = \binom{q}{1} \#1 + \binom{q}{2} \#2 + \dots + \binom{q}{q} \#q$ . Solving recursively we can now find  $\#q$ , i.e. if we know  $\#1$  and  $\#(\leq 2)$  then we can solve for  $\#2$ , then if we know these values as well as  $\#(\leq 3)$  then we can solve for  $\#3$ , and so on.

We can alternatively use the inclusion-exclusion principle to express  $\#q$  directly in terms of  $\#(\leq i)$ . For let  $X_i$  denote the sequences which do not contain the letter  $i$ . Because any letter  $j$  can substitute for any other letter  $i$  we have  $|X_i| = |X_j|$  for all  $i, j$ . Also

$$|X_i \cap X_j| = |X_{i'} \cap X_{j'}| \text{ for all } i, j, i', j', \text{ and so on.}$$

Now in this case, e.g. Comtet (1974, p.180), the inclusion-exclusion principle states that

$$\begin{aligned} |X_1 \cup X_2 \cup \dots \cup X_q| &= \binom{q}{1} |X_1| - \binom{q}{2} |X_1 \cap X_2| + \binom{q}{3} |X_1 \cap X_2 \cap X_3| - \dots \\ &\quad + (-1)^{q-1} |X_1 \cap X_2 \cap \dots \cap X_q|. \end{aligned}$$

Obviously  $|X_1 \cup X_2 \cup \dots \cup X_q| = \#(\leq q) - \#q$ , and  $|X_1| = \#(\leq q-1)$ , and

$|X_1 \cap X_2| = \#(\leq q-2)$ , and so on. Therefore

$$\begin{aligned} \#(\leq q) - \#q &= \binom{q}{1} \#(\leq q-1) - \binom{q}{2} \#(\leq q-2) + \dots + (-1)^{q-1} 0, \text{ so that} \\ \#q &= \#(\leq q) - \binom{q}{1} \#(\leq q-1) + \binom{q}{2} \#(\leq q-2) - \dots + (-1)^{q-1} \binom{q}{1} \#(\leq 1). \end{aligned} \tag{2.5}$$

Now suppose we are dealing with sequence structures, i.e. sequences for which merely a new permutation of the characters of the alphabet does not change the structure. With regard to the Polya and de Bruijn counting methods we are in a situation where  $G \times H = G \times S_q$ . Here

$$\text{it is obvious that } \#_S(\leq q) = \#_S 1 + \#_S 2 + \dots + \#_S q, \quad (2.6)$$

where the  $S$  attached to the hashes emphasizes the fact that we are dealing with structures.

$$\text{Obviously } \#_S(\leq q) = \#_S(\leq q-1) + \#_S q, \quad (2.7)$$

$$\text{so that } \#_S q = \#_S(\leq q) - \#_S(\leq q-1). \quad (2.8)$$

We shall now attempt to combine the above two situations and relate counts of sequence structures to counts of ordinary sequences. In particular, suppose we have a situation in

which  $\#_S q = \frac{\#q}{q!}$ , i.e. for the  $\#q$  ordinary sequences, every permutation of the characters of

the alphabet generates a different sequence. Further suppose that equation (2.5) holds. Then equation (2.7) becomes

$$\begin{aligned} \#_S(\leq q) &= \#_S(\leq q-1) + \#_S q \\ &= \#_S(\leq q-1) + \frac{\#q}{q!} \\ &= \#_S(\leq q-1) + \left( \frac{1}{q!} \right) \begin{pmatrix} \#(\leq q) - \binom{q}{1} \#(\leq q-1) + \binom{q}{2} \#(\leq q-2) - \\ \dots + (-1)^{q-1} \binom{q}{1} \#(\leq 1) \end{pmatrix}. \end{aligned}$$

Now consider this last equation. Obviously if  $i > k$  then the coefficient of  $\#(\leq i)$  in  $\#_S(\leq k)$  is 0.

Also, the coefficient of  $\#(\leq k)$  in  $\#_S(\leq k)$  is  $\frac{1}{k!}$  where  $\frac{1}{k!} = \left( \frac{1}{k!} \right) \left[ \binom{k}{0} a_{k+0, k} \right]$  and we

define  $a_{k+0, k} = 1$ .

The coefficient of  $\#(\leq k)$  in  $\#_S(\leq k+1)$  is

$$\begin{aligned}
\frac{1}{k!} \left[ \binom{k}{0} a_{k+0, k} \right] - \frac{1}{(k+1)!} \binom{k+1}{1} &= \frac{1}{(k+1)!} \left[ (k+1) \binom{k}{0} a_{k+0, k} - \binom{k+1}{1} \right] \\
&= \frac{1}{(k+1)!} \left[ \binom{k+1}{1} (a_{k+0, k} - 1) \right] \\
&= \frac{1}{(k+1)!} \left[ \binom{k+1}{1} a_{k+1, k} \right],
\end{aligned}$$

where we have defined  $a_{k+1, k} = a_{k+0, k} - 1$ .

The coefficient of  $\#(\leq k)$  in  $\#_S(\leq k+2)$  is

$$\begin{aligned}
\frac{1}{(k+1)!} \left[ \binom{k+1}{1} a_{k+1, k} \right] + \frac{1}{(k+2)!} \binom{k+2}{2} &= \frac{1}{(k+2)!} \left[ (k+2) \binom{k+1}{1} a_{k+1, k} + \binom{k+2}{2} \right] \\
&= \frac{1}{(k+2)!} \left[ \frac{(k+2)}{2} \binom{k+1}{1} 2a_{k+1, k} + \binom{k+2}{2} \right] \\
&= \frac{1}{(k+2)!} \left[ \binom{k+2}{2} (2a_{k+1, k} + 1) \right] \\
&= \frac{1}{(k+2)!} \left[ \binom{k+2}{2} a_{k+2, k} \right],
\end{aligned}$$

where we have defined  $a_{k+2, k} = 2a_{k+1, k} + 1$ .

In general, the coefficient of  $\#(\leq k)$  in  $\#_S(\leq k+m+1)$  is

$$\begin{aligned}
\frac{1}{(k+m)!} \left[ \binom{k+m}{m} a_{k+m, k} \right] + \frac{1}{(k+m+1)!} \left[ (-1)^{m+1} \binom{k+m+1}{m+1} \right] \\
&= \frac{1}{(k+m+1)!} \left[ (k+m+1) \binom{m+1}{m+1} \binom{k+m}{m} a_{k+m, k} + (-1)^{m+1} \binom{k+m+1}{m+1} \right] \\
&= \frac{1}{(k+m+1)!} \left[ \binom{k+m+1}{m+1} ((m+1)a_{k+m, k} + (-1)^{m+1}) \right] \\
&= \frac{1}{(k+m+1)!} \left[ \binom{k+m+1}{m+1} a_{k+m+1, k} \right] \\
&= \frac{1}{(k+m+1)!} \left[ \binom{k+m+1}{k+m+1-k} a_{k+m+1, k} \right],
\end{aligned}$$

where, in general, we have defined  $a_{k+m+1, k} = (m+1)a_{k+m, k} + (-1)^{m+1}$ .

Drawing all of this information together, and replacing  $k$  with  $i$  and  $k+m+1$  with  $q$ , we have

$$\#_S(\leq q) = \sum_{i=1}^q \frac{1}{q!} \binom{q}{q-i} a_{q,i} \#(\leq i). \quad (2.9)$$

Now in the problème des rencontres, a derangement is a permutation in which no element remains in its original position, e.g. Comtet (1974, p. 180). If  $D_{m,0}$  denotes the number of derangements then  $D_{m+1,0} = (m+1)D_{m,0} + (-1)^{m+1}$  and  $D_{0,0} = 1$ , e.g. Riordan (1958, p.59). Thus the numbers  $a_{k+m,k}$  are identical to the numbers of derangements  $D_{m,0}$ .

Furthermore, if  $D_{m,k}$  denotes the number of permutations in which exactly  $k$  elements remain in their original positions then, again from Riordan (1958, p.59), we have

$$D_{m,k} = \binom{m}{k} D_{m-k,0}.$$

Substituting  $D$ 's for  $a$ 's in equation 2.9 we obtain:

$$\begin{aligned} \#_S(\leq q) &= \sum_{i=1}^q \frac{1}{q!} \binom{q}{q-i} a_{q,i} \#(\leq i) \\ &= \frac{1}{q!} \sum_{i=1}^q \binom{q}{q-i} D_{q-i,0} \#(\leq i) \\ &= \frac{1}{q!} \sum_{i=1}^q D_{q,i} \#(\leq i). \end{aligned}$$

Thus we have established the following theorem.

**Theorem 2.15.** If  $\#_S q = \frac{\#q}{q!}$  and if

$$\#q = \#(\leq q) - \binom{q}{1} \#(\leq q-1) + \binom{q}{2} \#(\leq q-2) - \dots + (-1)^{q-1} \binom{q}{1} \#(\leq 1) \text{ then}$$

$$\#_S(\leq q) = \frac{1}{q!} \sum_{i=1}^q D_{q,i} \#(\leq i).$$

## 2.9 Evaluating the formulae

Having all the formulae without looking at the numbers is like drinking hot water without adding the coffee.

We shall let  $G \times H$  denote the combined effects of  $G$  and  $H$ , where  $G$  is a group of permutations acting on the positions within a sequence, and  $H$  is a group of permutations acting on the alphabetical characters allocated to the positions within a sequence. We are mainly interested in the case  $H = S_q$  where  $S_q$  is the symmetric group, i.e. the group of all possible permutations of the alphabet.

In the above sections we described how to compute numbers of sequences with exactly  $q$  different characters once we know the number of sequences with  $\leq q$  characters. We also described how the Möbius inversion formula can be used to compute numbers of primitive sequences no matter whether they contain exactly  $q$  different characters or  $\leq q$  characters. Maple scripts were used to evaluate all of the formulae for  $1 \leq n \leq 31$  and for  $1 \leq q \leq 6$ . The results are presented in Appendix A. When one considers that the present chapter has been very strongly influenced by the papers of Gilbert and Riordan (1961) and Titsworth (1964) it is not surprising that parts of these tables have already occurred in the literature. In fact, Sloane's absolutely marvellous On-Line Encyclopedia of Integer Sequences (1998) was consulted for ascertaining whether or not any column of any table had previously been registered with Sloane. All columns which had previously been registered are shaded in the tables. Therefore these numbers, or the formulae which generate them, have almost certainly been previously published. On the other hand, non-shaded columns have, almost certainly, not been previously published.

### 2.9.1 Special comments concerning $I_n \times I_q$

$I_n$  is the "do nothing" group of transformations so here we merely count the total number of possible sequences using an alphabet of  $q$  characters. For sequences of length  $n$  we have already shown that the cycle index is  $P_{I_n}(x_1, \dots, x_n) = x_1^n$  and so the corresponding pattern

inventory is  $\left( \sum_{i=1}^q c_i \right)^n$ . We obtain the total number of sequences with  $\leq q$  characters by

substituting 1 for each of the  $c_i$ 's. This yields  $q^n$  as expected.

The "cycle index" for the total number of primitive sequences is  $\sum_{d|n} \mu(d) x_1^{n/d}$  yielding

$\sum_{d|n} \mu(d) q^{n/d}$  for the total number of primitive sequences.

Now it is well-known, e.g. Comtet (1974, p.204), that  $S(n, q)$ , a Stirling number of the second kind, is the number of ways of partitioning a set of  $n$  objects into  $q$  nonempty subsets. Without loss of generality we consider the  $n$  objects to be the first  $n$  positive integers, and so each nonempty subset of this set of  $n$  objects can be interpreted as a nonempty subset of the  $n$  positions in a finite sequence of length  $n$ . A different character of an alphabet can be assigned to each of the  $q$  nonempty subsets, thereby generating a  $q$ -ary sequence. The  $q$  characters can be assigned to the  $q$  subsets in  $q!$  ways, so  $q! S(n, q)$  is the total number of finite sequences of length  $n$  which contain exactly  $q$  different characters.

We shall also need the following recurrence relation for Stirling numbers, e.g. Comtet (1974, p.208):

$$S(n+1, q) = q S(n, q) + S(n, q-1).$$

Using equation (2.5) to count the number of sequences with exactly  $q$  different characters we have  $\#q = q^n - \binom{q}{1}(q-1)^n + \binom{q}{2}(q-2)^n - \dots + (-1)^{q-1} \binom{q}{1} 1^n$ . Also,  $\#q = q! S(n, q)$  so we can combine these two formulae to confirm the well-known identity, e.g. Comtet (1974, p. 240),

$$q!S(n, q) = q^n - \binom{q}{1}(q-1)^n + \binom{q}{2}(q-2)^n - \dots + (-1)^{q-1} \binom{q}{1} 1^n.$$

From the Möbius inversion formula we also conclude that the total number of primitive

sequences which contain exactly  $q$  different characters is  $\sum_{d|n} \mu(d) q! S\left(\frac{n}{d}, q\right)$ .

### 2.9.2 Special comments concerning $I_n \times S_q$

Here we are counting sequence structures such that permuting the characters of the alphabet does not change the structure. Under the action of  $I_n$  alone, there are  $q!$  sequences all with

the same structure so  $\#_S q = \frac{\#q}{q!}$ . Obviously the conditions of Theorem 2.15 apply and we

have

$$\#_S (\leq q) = \frac{1}{q!} \sum_{i=1}^q D_{q,i} \#(\leq i) = \frac{1}{q!} \sum_{i=1}^q D_{q,i} i^n. \quad (2.10)$$

Now  $\#k = k! S(n, k)$  so we have here  $\#_S k = \frac{\#k}{k!} = S(n, k)$  and substituting this into equation

(2.6) gives

$$\#_S (\leq q) = \sum_{k=1}^q S(n, k). \quad (2.11)$$

Combining equations (2.10) and (2.11) proves the following theorem which relates Stirling numbers of the second kind to numbers of derangements.

**Theorem 2.16.**  $\sum_{k=1}^q S(n, k) = \frac{1}{q!} \sum_{i=1}^q D_{q,i} i^n.$

As far as I can ascertain, the identity in Theorem 2.16 has not been previously published.

We have already shown above that the number of sequence structures which contain exactly

$q$  different characters is simply  $S(n, q)$ . We again use the Möbius inversion formula to obtain the number of primitive sequence structures which contain exactly  $q$  different characters. It will be  $\sum_{d|n} \mu(d)S\left(\frac{n}{d}, q\right)$ .

## 2.10 Palindromes

If the reverse of either a finite sequence or a periodic sequence is identical to the original sequence then the sequence is said to be a palindrome. Thus with regard to finite sequences,  $A = (a, b, c, c, b, a)$  is a palindrome but  $B = (b, c, c, b, a, a)$  is not a palindrome, even though it is a cyclic shift of the original sequence. However, with regard to periodic sequences, if  $S$  is a finite sequence representation of a periodic sequence then any cyclic shift of  $S$  is also a representation of the same periodic sequence. Thus  $A$  and  $B$  above are both finite sequence representations of the same palindromic periodic sequence.

A primitive palindrome is one whose sequence is primitive.

### 2.10.1 A special method for counting palindromes

Consider an arbitrary set  $C$  of finite sequences all of which have a particular desired property. Let  $\#C$  denote the total number of sequences in the set,  $\#P$  denote the total number of palindromes in the set, and  $\#nP$  denote the total number of nonpalindromes in the set. Suppose there is an equivalence relation defined on  $C$  such that there is exactly one sequence in an equivalence class if the class contains a palindrome, and there are exactly two sequences in an equivalence class if the class does not contain any palindromes. Obviously the equivalence relation we have in mind is based on reversals of sequences. We use  $\#R$  to denote the number of equivalence classes. Clearly

$$\#C = \#P + \#nP \tag{2.12}$$

$$\text{and } \#R = \#P + \frac{1}{2}\#nP. \tag{2.13}$$

$$\text{From equation (2.12) we get } \#nP = \#C - \#P, \tag{2.14}$$

and substituting in equation (2.13) gives  $\#R = \frac{1}{2}(\#C + \#P)$ , (2.15)

or equivalently  $\#P = 2\#R - \#C$ . (2.16)

These latter equations provide trivially easy methods for using palindromes to count numbers of equivalence classes under reversal, and vice versa, provided the assumptions are fulfilled.

Similar simple arguments apply to periodic sequences. In this case C will be a set of finite sequences such that each sequence represents a cyclic equivalence class. #R will be the total number of dihedral equivalence classes.

### **2.10.2 Counting finite palindromes and finite palindromic structures**

If  $S = (s_1, s_2, \dots, s_n)$  is a finite sequence which is also a palindrome then

$(s_1, s_2, \dots, s_n) = (s_n, s_{n-1}, \dots, s_2, s_1)$  so that  $s_1 = s_n$ ,  $s_2 = s_{n-1}$ , and so on. Thus S has the form  $TR(T)$  if  $n$  is even, and the form  $TaR(T)$  if  $n$  is odd, for some character  $a$  of the alphabet.

#### 2.10.2.1 Number of finite palindromes which contain $\leq q$ different characters

Using the above observation, if  $n$  is even then the number of finite palindromes with  $q$  or fewer different characters is given by  $\#(\leq q) = q^{n/2}$ .

Now suppose  $n$  is odd, so that the palindrome has form  $TaR(T)$ . Sequence T can be

generated in  $q^{(n-1)/2}$  ways and  $a$  can be chosen in  $q$  different ways so that

$$\#(\leq q) = q \times q^{(n-1)/2} = q^{(n+1)/2}.$$

#### 2.10.2.2 Number of finite palindromes which contain exactly $q$ different characters

If  $n$  is even then  $\#q = q!S\left(\frac{n}{2}, q\right)$ .

Now suppose  $n$  is odd so that the palindrome has form  $TaR(T)$ . If T has  $q$  different

characters then there are  $q$  different possible values that  $a$  can assume. If  $T$  has  $q-1$  different characters then  $a$  can assume only one possible value. If  $T$  has fewer than  $q-1$  different characters then no such palindrome exists. Therefore

$$\begin{aligned}\#q &= q \times q! S\left(\frac{n-1}{2}, q\right) + \binom{q}{q-1} (q-1)! S\left(\frac{n-1}{2}, q-1\right) \\ &= q \times q! S\left(\frac{n-1}{2}, q\right) + q(q-1)! S\left(\frac{n-1}{2}, q-1\right) \\ &= q! \left[ q S\left(\frac{n-1}{2}, q\right) + S\left(\frac{n-1}{2}, q-1\right) \right] \\ &= q! S\left(\frac{n+1}{2}, q\right).\end{aligned}$$

#### 2.10.2.3 Number of finite palindromic structures which contain exactly $q$ different characters

If  $n$  is even then  $\#_S q = S\left(\frac{n}{2}, q\right)$ .

If  $n$  is odd then either  $a$  occurs in  $T$  or it does not. Because we are dealing with structures, if  $a$  is not in  $T$  then it does not matter at all which  $q-1$  characters occur in  $T$  because we always have the same structure. Thus we obtain

$$\#_S q = q \times S\left(\frac{n-1}{2}, q\right) + S\left(\frac{n-1}{2}, q-1\right) = S\left(\frac{n+1}{2}, q\right).$$

#### 2.10.2.4 Number of finite palindromic structures which contain $\leq q$ different characters

Here we can simply combine equation (2.6) with the results immediately above. Thus if  $n$  is even then  $\#_S (\leq q) = \sum_{k=1}^q S\left(\frac{n}{2}, k\right)$ , and if  $n$  is odd then  $\#_S (\leq q) = \sum_{k=1}^q S\left(\frac{n+1}{2}, k\right)$ .

#### 2.10.2.5 Further comments about finite palindromes

Some actual counts of finite palindromes for small values of  $n$  and  $q$  are presented in Appendix A.

As an alternative to the above formulae, we can use equation (2.16) and the tables of counts for  $I_n \times I_q$  and  $R_n \times I_q$  to count finite palindromes, but not finite palindromic structures.

This latter failure is due to the fact that, with regard to structures, there are not necessarily two different structures in an equivalence class under the action of  $R_n \times S_q$  even if the class does not contain a palindrome! For example, under  $R_4 \times S_3$ , the sequence  $(a,b,c,a)$  is the sole member of a class, in the sense that its reverse  $(a,c,b,a)$  has the same structure, but this sequence is not a palindrome. Compare this with the class  $\{(a,a,a,b), (b,a,a,a)\}$  which has two representatives, each of different structure.

### 2.10.3 Counting periodic palindromes

When working from first principles I have managed to arrive at the formulae for counts of periodic palindromes, but my method is long and arduous. I wonder if there is a slick direct method. In any case, if one accepts the cycle indices for cyclic and dihedral equivalence classes then equation (2.16) provides the solution rather quickly.

Recall that equation (2.16) states that  $\#P = 2\#R - \#C$  where  $\#P$  is the number of palindromes,  $\#R$  is the number of dihedral equivalence classes and  $\#C$  is the number of cyclic equivalence classes. We can even invoke our little trick of using cycle indices initially, instead of counts, in order to obtain a "cycle index" for periodic palindromes. We

$$\text{have } P_{D_n}(x_1, \dots, x_n) = \frac{1}{2n} \sum_{d|n} \varphi(d) x_d^{n/d} + \frac{1}{2} x_1 x_2^{(n-1)/2} \text{ for } n \in O,$$

$$P_{D_n}(x_1, \dots, x_n) = \frac{1}{2n} \sum_{d|n} \varphi(d) x_d^{n/d} + \frac{1}{4} x_1^2 x_2^{(n-2)/2} + \frac{1}{4} x_2^{n/2} \text{ for } n \in E, \text{ and the cycle index for}$$

$$C_n \text{ is } \frac{1}{n} \sum_{d|n} \varphi(d) x_d^{n/d}. \text{ Therefore the "cycle index" for periodic palindromes is } x_1 x_2^{(n-1)/2} \text{ for}$$

$$n \in O, \text{ and } \frac{1}{2} x_1^2 x_2^{(n-2)/2} + \frac{1}{2} x_2^{n/2} \text{ for } n \in E. \text{ Using the pattern inventory associated with}$$

these "cycle indices" we conclude that the numbers of periodic palindromes are  $q^{(n+1)/2}$  for

$$n \in O, \text{ and } \frac{1}{2} q^2 q^{(n-2)/2} + \frac{1}{2} q^{n/2} = \frac{1}{2} q q^{n/2} + \frac{1}{2} q^{n/2} = \frac{1}{2} q^{n/2}(q+1) \text{ for } n \in E.$$

## 2.10.4 Counting periodic palindromic structures

I have not been able to find any formulae for counting periodic palindromic structures. The actual counts in the tables in Appendix A were obtained from computer searches.

## 2.11 Discussion

Fine (1958), whom I have not tried to understand, laboriously computed and tabulated primitive and nonprimitive counts of periodic sequences with  $\leq q$  characters under the action of  $C_n \times S_2$  and  $C_n \times S_3$ .

Gilbert and Riordan (1961) used the methods of Polya and de Bruijn to count primitive and nonprimitive sequences with  $\leq 2$  characters under the actions of  $C_n \times I_2$ ,  $C_n \times S_2$ ,  $D_n \times I_2$  and  $D_n \times S_2$ . This paper should probably be regarded as the classic paper with respect to counting equivalence classes of sequences.

It seems fair to say that Gilbert and Riordan (1961) was also the inspiration for Titsworth (1964) which clearly laid the foundation for the present chapter. Titsworth tabulated primitive and nonprimitive sequences with  $\leq 2$  characters under the actions of  $E_n \times I_2$ .

None of the afore-mentioned authors considered the calculation of sequences which contain exactly  $q$  different letters.

Apart from the periodic palindromic structures, the tables in Appendix A are subject to two possible sources of error, viz. errors in the mathematical formulae themselves and errors in the Maple scripts which I wrote to evaluate the formulae. There are a few different reasons why I believe the tables have no errors:

- (1) Sequences which I expected to find in Sloane (1998) were actually there.
- (2) The Maple output was fed directly into my word processor and I did not use any manual transcription.
- (3) As discussed in the final chapter, I also wrote C programs which explicitly searched for a representative of each of the equivalence classes. For small values of  $n$  and  $q$  it was feasible

to run these searches. For at least six different combinations of  $n$  and  $q$  in every table, the number of sequences found by computer search agreed exactly with the tabulated number evaluated using Maple. Of course this also provided a useful cross-check on the accuracy of my C coding.

The periodic palindromic structures could only be obtained using computer searches. As an internal check, I verified that the numbers of primitive and nonprimitive palindromes were compatible using the Möbius inversion formula.

With regard to the tables in Appendix A, under the action of  $G \times I_q$ , i.e. tables other than tables of structures, the number of primitive sequences with  $\leq 2$  characters must equal the number of primitive sequences with exactly 2 different characters except for the case  $n=1$  in which case the latter must obviously be zero. When checking Sloane (1998) for the presence of these sequences I ignored the case  $n=1$  and merely considered sequence lengths or periods  $> 1$ .

The intention in this chapter has been to count equivalence classes of finite and periodic sequences with certain properties. The common analogy with necklaces has already been mentioned. When one considers those sequences in Appendix A which also appear in Sloane (1998), and if one reads the references in Sloane, then it is apparent that the tables of counts have other applications which include shift register sequences, irreducible polynomials, graph coverings, and even wire bending ( $R_n \times S_3$ ).

It would also be interesting to count equivalence classes for which each alphabetical character occurs an equal number of times in a sequence, for suitable values of  $n$ .

## References

- van Aardenne-Ehrenfest, T. and de Bruijn, N. G. (1951). Circuits and trees in oriented linear graphs. *Simon Stevin* 28: 203-217.
- Adams, W. T. and Birkes, D. S. (1990). Mating patterns in seed orchards. Proc. 20th South. Forest Tree Improvement Conf., Charleston, South Carolina, June 27-29, 1989: 75-86.
- Adobe Systems Inc. (1985). *Postscript Language Tutorial and Cookbook*. Addison-Wesley Publishing Company, Inc.
- Agroforestry Systems (1982). 1(1): 7-12.
- Antonovics, J. and Fowler, N. L. (1985). Analysis of frequency and density effects on growth in mixtures of *Salvia splendens* and *Linum grandiflorum* using hexagonal fan designs. *Journal of Ecology* 73: 219-234.
- Baumert, L. D. (1971). *Cyclic Difference Sets*. Lecture Notes in Mathematics 182. Springer-Verlag.
- Bell, G. D. and Fletcher, A. M. (1978). Computer organised orchard layouts (COOL) based on the permuted neighbourhood design concept. *Silvae Genetica* 27: 223-225.
- Bleasdale, J. K. A. (1967). Systematic designs for spacing experiments. *Expl Agric.* 3: 73-85.
- Bocquet, M. (1953). Note sur l'expérience de densité marchal appliquée à l'hévéaculture. Archief voor de Rubber Cultur (Archives of Rubber Cultivation) 30: 194-199.
- Boffey, T. B. and Veevers, A. (1979). On the non-existence of balanced designs for two-variety competition experiments. *Utilitas Mathematica* 16: 131-143.

Briscoe, C. B. (1989). Field Trials Manual for Multipurpose Tree Species. Multipurpose Tree Species Network Research Series, Winrock International Institute for Agricultural Development.

de Bruijn, N. G. (1959). Generalization of Polya's fundamental theorem in enumerative combinatorial analysis. *Nederl. Akad. Wetensch. Proc. Ser. A* 62 = *Indag. Math.* 21: 59-69.

de Bruijn, N. G. (1964). Polya's theory of counting. In E. F. Beckenbach (ed.), *Applied Combinatorial Mathematics*: 144-184. John Wiley & Sons (New York).

van Buijtenen, J. P. (1971). Seed orchard design, theory and practice. Proc. of the Eleventh Conference on Southern Forest Tree Improvement, Atlanta, Georgia, June 15-16, 1971: 197-206.

Chakravarty, G. N. and Bagchi, S. K. (1993). A computer program for permuted neighbourhood seed orchard design. *Silvae Genetica* 41: 1-5.

Chartrand, G. and Oellermann, O. R. (1993). *Applied and Algorithmic Graph Theory*. McGraw-Hill, Inc.

Cochran, W. G. and Cox, G. M. (1957). *Experimental Designs*. Second edition. John Wiley & Sons, Inc.

Comtet, L. (1974). *Advanced Combinatorics*. D. Reidel Publishing Company.

Cormack, R. M. (1979). Spatial aspects of competition between individuals. In R. M. Cormack and J. K. Ord (eds.), *Spatial and Temporal Analysis in Ecology*, Fairland, Maryland, USA: International Co-operative Publishing House.

Coxeter, H. S. M. (1961). *Introduction to Geometry*. John Wiley & Sons.

de la Cruz, R. E. and Vergara, N. T. (1987). Protective and ameliorative roles of agroforestry: an overview. In N. T. Vergara and N. D. Briones (eds.), *Agroforestry in the Humid Tropics: its protective and ameliorative roles to enhance productivity and sustainability*, Environment and Policy Institute, East-West Center, Honolulu, Hawaii, USA: 3-30.

Davis, P. J. (1979). *Circulant Matrices*. John Wiley & Sons, Inc.

Day, R. and Street, A. P. (1982). Sequential binary arrays I. The square grid. *J. Combin. Theory A* 32: 35-52.

Diggle, P. J. (1983). *Statistical Analysis of Spatial Point Patterns*. Academic Press Inc. (London) Ltd.

Domb, C. (1960). On the theory of cooperative phenomena in crystals (continued). *Advances in Physics* 9: 245-361.

Donald, C. M. (1958). The interaction of competition for light and for nutrients. *Australian Journal of Agricultural Research* 9(4): 421-435.

Donald, C. M. (1963). Competition among crop and pasture plants. *Advances in Agronomy* 15: 1-118.

Draper, N. R. and Smith, H. (1981). *Applied Regression Analysis*. Second edition. John Wiley & Sons.

Dyson, W. G. and Freeman, G. H. (1968). Seed orchard designs for sites with a constant prevailing wind. *Silvae Genetica* 17: 12-15.

Erickson, V. J. and Adams, W. T. (1989). Mating success in a coastal Douglas-fir seed orchard as affected by distance and floral phenology. *Can. J. For. Res.* 19: 1248-1255.

Fasoulas, A. C. (1988). *The Honeycomb Methodology of Plant Breeding*. Dept. of Genetics and Plant Breeding, Aristotelian University of Thessaloniki, Greece.

Fine, N. J. (1958). Classes of periodic sequences. *Illinois J. Math.* 2: 285-302.

Fitter, A. H. (1976). Effects of nutrient supply and competition from other species on root growth of *Lolium perenne* in soil. *Plant and Soil* 45(1): 177-189.

Fleischner, H. (1983). Eulerian graphs. In L. W. Beineke and R. J. Wilson (eds.), *Selected Topics in Graph Theory 2*, Academic Press (London): 17-53.

Freeman, G. H. (1967). The use of cyclic balanced incomplete block designs for directional seed orchards. *Biometrics* 23: 761-778.

Freeman, G. H. (1969). The use of cyclic balanced incomplete block designs for non-directional seed orchards. *Biometrics* 25: 561-571.

Gates, D. J. (1980). Competition between two types of plants with specified neighbour configurations. *Mathematical Biosciences* 48: 195-209.

Giertych, M. M. (1965). Systematic lay-outs for seed orchards. *Silvae Genetica* 14: 91-94.

Giertych, M. M. (1971). Systematic lay-outs for seed orchards. *Silvae Genetica* 20: 137-138.

Giertych, M. (1975). Seed orchard designs. In R. Faulkner (ed.), *Seed Orchards*, Forestry Commission Bulletin No. 54 (London): 25-37.

Gilbert, E. N. and Riordan, J. (1961). Symmetry types of periodic sequences. *Illinois J. Math.* 5: 657-665.

Golomb, S. W. (1954). Checker boards and polyominoes. *American Mathematical Monthly* 61: 675-682.

Golomb, S. W. (1967). *Shift Register Sequences*. Holden-Day, Inc. (San Francisco).

Good, I. J. (1946). Normal recurring decimals. *J. London Math. Soc.* 21: 167-169.

Grünbaum, B. and Shephard, G. C. (1980). Satins and twills: An introduction to the geometry of fabrics. *Mathematics Magazine* 53(3): 139-161.

Hall, P. (1935). On representatives of subsets. *J. London Math. Soc.* 10: 26-30.

Hall, R. L. (1974). Analysis of the nature of interference between plants of different species. I. Concepts and extension of the de Wit analysis to examine effects. *Aust. J. Agric. Res.* 25: 739-747.

Harper, J. L. (1961). Approaches to the study of plant competition. *Symp. Soc. Exp. Biol.* 15: 1-39.

Hart, R. D. (1975). A bean, corn and manioc polyculture cropping system. II. A comparison between the yield and economic return from monoculture and polyculture cropping systems. *Turrialba* 25(4): 377-384.

Hendrickson Jr., J. A. (1995). On the enumeration of rectangular  $(0,1)$ -matrices. *Journal of Statistical Computation and Simulation* 51: 291-313.

Hoskins, J. A., Hoskins, W. D., Street, A. P. and Stanton, R. G. (1982). Some elementary isonemal binary matrices. *Ars Combinatoria* 13: 3-38.

Hung, S. H. Y. and Mendelsohn, N. S. (1974). Handcuffed designs. *Aequationes Mathematicae* 11: 256-266.

Hutchinson, J. P. and Wilf, H. S. (1975). On Eulerian circuits and words with prescribed adjacency patterns. *Journal of Combinatorial Theory A* 18: 80-87.

Huxley, P. A. (1983). The role of trees in agroforestry: some comments. In P. A. Huxley, (ed.), *Plant Research and Agroforestry*, ICRAF, Nairobi, Kenya: 257-270.

Huxley, P. A. (1985). Systematic designs for field experimentation with multipurpose trees. *Agroforestry Systems* 3: 197-207.

Huxley, P., Darnhofer, T., Pinney, A., Akunda, E. and Gatama, D. (1989). The tree/crop interface: a project designed to generate experimental methodology. *Agroforestry Abstracts* 2(4): 127-145.

Huxley, P., Mead, R. and Ngugi, D. (1987). National Agroforestry Research Proposals for Southern Africa AFRENA. International Council for Research in Agroforestry (ICRAF): Nairobi, Kenya.

ICAR/ICRAF (1987). Report of Working Committee 4 on experimental approaches and plans. ICAR/ICRAF agroforestry training/workshop on agroforestry research, Sept 16-30th, 1986, Hyderabad, India.

ICRAF (1988). Proceedings of an ICRAF mini-workshop on experimental design, Nairobi, Kenya, 7-11 June, 1988. (Ed. J. H. Roger).

ICRAF/CFI (1983). Methodology for the exploration and assessment of multipurpose trees: Section four, part 4a. Introduction (plus supplement on MPT mixed cropping trials). (Ed. P. A. Huxley).

ICRAF/CFI (1985). Methodology for the exploration and assessment of multipurpose trees: Section six, part 6c. Glossary of terms used in agroforestry.

Joyce, G. S. (1973). On the simple cubic lattice Green function. Philosophical Transactions of the Royal Society of London A 273: 583-610.

Keddy, P. A. (1989). *Competition*. Population and Community Biology Series, Chapman and Hall.

Kernighan, B. W. and Ritchie, D. M. (1978). *The C Programming Language*. Prentice-Hall, Inc.

Klaehn, F. U. (1960). Seed orchard classification. J. Forestry 58: 355-360.

La Bastide, J. G. A. (1967). A computer program for the layouts of seed orchards. Euphytica 16: 321-323.

Langner, W. (1953). Die Klonanordnung in Samenplantagen. Z. Forstgenetik 2: 119-121.

Langner, W. and Stern, K. (1955). Versuchstechnische Probleme bei der Anlage von Klonplantagen. Z. Forstgenetik 4: 81-88.

Langton, S. (1990). Avoiding edge effects in agroforestry experiments; the use of neighbour-balanced designs and guard areas. Agroforestry Systems 12: 173-185.

Larsen, C. S. (1956). *Genetics in Silviculture*. (Translated by M. L. Anderson). Oliver and Boyd (Edinburgh).

Las Vergnas, M. (1990). An upper bound for the number of Eulerian orientations of a regular graph. Combinatorica 10 (1): 61-65.

Lin, C. and Morse, P. M. (1975). A compact design for spacing experiments. Biometrics 31: 661-671.

Liskovec, V. A. (1971). The number of Eulerian digraphs and regular tournaments. *Vesci Akad. Navuk Belaruskaya Ser. Fiz.-Mat. Navuk* 1: 22-27.

Liyanage, M. de S., Tejwani, K. G. and Nair, P. K. R. (1984). Intercropping under coconuts in Sri Lanka. *Agroforestry Systems* 2: 215-228.

Macdonald, S. O. and Street, A. P. (1979). Balanced binary arrays II: The triangular grid. *Ars Combinatoria* 8: 65-84.

Malac, B. F. (1962). Shifting-clone design for a superior tree seed orchard. *Woodland Res. Notes*, Union Bag-Camp Paper Corporation, Savannah, Georgia, U.S.A. 14: 1-3.

Maple V (1981-1993). Release 2.0a for Microsoft Windows. University of Waterloo.

Marsh, P. L. (1985). A flexible computer algorithm for designing seed orchards. *Silvae Genetica* 34: 22-26.

Martin, F. B. (1973). Beehive designs for observing variety competition. *Biometrics* 29: 397-402.

McKay, B. D. (1983). Application of a technique for labelled enumeration. *Congress. Numer.* 40: 207-221.

Mead, R. (1979). Competition experiments. *Biometrics* 35: 41-54.

Mead, R. (1997). Personal communication of 10 November 1997. University of Reading, England.

Mead, R. and Riley, J. (1981). A review of statistical ideas relevant to intercropping research. *J. R. Statist. Soc. A* 144(4): 462-509.

Mead, R. and Stern, R. D. (1980). Designing experiments for intercropping research. *Expl Agric.* 16: 329-342.

Mendelsohn, N. S. (1996). Mendelsohn designs. In C. J. Colbourn and J. H. Dinitz (eds.), *The CRC Handbook of Combinatorial Designs*, CRC Press, Inc.: 388-393.

Mirsky, L. (1955). *An Introduction to Linear Algebra*. Oxford University Press.

Morgan, J. P. (1988). Balanced polycross designs. *J. R. Statist. Soc. B* 50(1): 93-104.

Morgan, J. P. (1990). Some series constructions for two-dimensional neighbor designs. *Journal of Statistical Planning and Inference* 24: 37-54.

Nair, P. K. R. (1983). Agroforestry with coconuts and other tropical plantation crops. In: Huxley, P. A. (ed.), *Plant Research and Agroforestry*, ICRAF, Nairobi, Kenya: 79-102..

Nair, P. K. R. (1989). Classification of agroforestry systems. In P. K. R. Nair (ed.), *Agroforestry Systems in the Tropics*, Kluwer Academic Publishers.

Nelder, J. A. (1962). New kinds of systematic designs for spacing experiments. *Biometrics* 18: 283-307.

Nester, M. R. (1994a). Eulerian orientations and circuits of complete bipartite graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing* 16: 3-17.

Nester, M. R. (1994b). HAHA designs. *Australasian Journal of Combinatorics* 9: 261-274.

Nester, M. R. (1994c). Modulo tile constructions for systematic seed orchard designs. *Silvae Genetica* 43: 312-321.

Nester, M. R. (1994d). A Compendium of Designs and Plant Arrangements for the Study and Control of Plant Interactions. Unpublished report. Queensland Forestry Research Institute, Gympie, Australia.

Nester, M. R. (1997). Sequential arrays. *Utilitas Mathematica* 51: 97-117.

Nester, P. M. (1991). Personal communication. Gympie, Australia.

Neumann, P. M., Stoy, G. A. and Thompson, E. C. (1994). *Groups and Geometry*. Oxford University Press.

Nijenhuis, A. and Wilf, H. S. (1975). *Combinatorial Algorithms*. Academic Press.

Niven, I. and Zuckerman, H. S. (1972). *An Introduction to the Theory of Numbers*. John Wiley & Sons Inc. (New York).

Oates-Williams, S. and Street, A. P. (1979). Balanced binary arrays III: The hexagonal grid. *J. Austral. Math. Soc. (Series A)* 28: 479-498.

Patterson, H. D. (1964). Theory of cyclic rotation experiments. *J. R. Statist. Soc. B* 26(1): 1-45 (including discussion).

Pearce, S. C. (1983). *The Agricultural Field Experiment*. John Wiley & Sons.

Petersen, R. G. (1985). *Design and Analysis of Experiments*. Marcel Dekker: New York.

Polya, G. and Read, R. C. (1987). *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*. Springer-Verlag (New York).

Praeger, C. E. and Nilrat, C. K. (1987). Sequential binary arrays and circulant matrices. *J. Austral. Math. Soc. A* 42: 330-348.

Praeger, C. E. and Street, A. P. (1983). Characterisation of some sparse binary sequential arrays. *Aequationes Mathematicae* 26: 54-58.

Putnam, D. H., Herbert, S. J. and Vargas, A. (1985). Intercropped corn-soyabean density studies. I. Yield complementarity. *Expl Agric.* 21: 41-51.

Rao, M. R., Kamara, C. S., Kwesiga, F. and Duguma, B. (1990). Methodological issues for research on improved fallows. *Agroforestry Today* 2(4): 8-12.

Rao, M. R., Sharma, M. M. and Ong, C. K. (1990). A study of the potential of hedgerow intercropping in semi-arid India using a two-way systematic design. *Agroforestry Systems* 11: 243-258.

Read, R. C. (1970). Graph theory algorithms. In B. Harris, (ed.), *Graph Theory and its Applications*, Academic Press (New York): 51-78.

Riitters, K. H., Stanton, B. J. and Walkup, R. H. (1989). Factor levels for density comparisons in the split-block spacing design. *Can. J. For. Res.* 19: 1625-1628

Riordan, J. (1958). *An Introduction to Combinatorial Analysis*. John Wiley & Sons (New York).

Robinson, P. J. (1986). Balanced cyclic binary arrays. *Ars Combinatoria* 21: 189-199.

Ryan, P. (1991). Personal communication. Queensland Forestry Research Institute, Gympie, Australia.

Sakai, K.-I. (1957). Studies on competition in plants. VII. Effect on competition of a varying number of competing and non-competing individuals. *Journal of Genetics* 55(2): 227-234.

Sanchez, P. A. (1995). Science in agroforestry. *Agroforestry Systems* 30: 5-55.

Schrijver, A. (1983). Bounds on the number of Eulerian orientations. *Combinatorica* 3 (3-4): 375-380.

Shelke, V. B. (1977). Studies on crop geometry in dryland intercrop systems. Ph. D thesis, Marathwada Agricultural University, Parbhani, Maharashtra, India.

Shen, H.-H., Rudin, D. and Lindgren, D. (1981). Study of the pollination pattern in a Scots pine seed orchard by means of isozyme analysis. *Silvae Genetica* 30: 7-15.

Singh, S. B., Pramod Kumar, and Prasad, K. G. (1989). Interference between two species in mixed seeding system. *Indian Forester* 115(12): 884-890.

Singh, R. P., Ong, C. K. and Saharan, N. (1989). Above and below ground interactions in alley-cropping in semi-arid India. *Agroforestry Systems* 9: 259-274.

Sloane, N. J. (1998). *The On-Line Encyclopedia of Integer Sequences*. AT&T Research, Murray Hill NJ 07974, USA, njas@research.att.com.

Smith, C. A. B. and Tutte, W. (1941). On unicursal paths in a network of degree 4. *Amer. Math. Monthly* 48: 233-237.

Smith, J. H. G. (1978). Design factors from Nelder and other spacing trials to age 20. *The Commonwealth Forestry Review* 57(2) No. 172: 109-119.

Snaydon, R. W. (1979). A new technique for studying plant interactions. *Journal of Applied Ecology* 16: 281-286.

Srinivasan, V. M. and Caulfield, J. (1989). Agro-forestry land management system in developing countries - an overview. *The Indian Forester* 115(2): 57-70.

Street, A. P. (1982). A survey of neighbour designs. *Congressus Numerantium* 34: 119-155.

Street, A. P. and Day, R. (1982). Sequential binary arrays II: Further results on the square grid. Combinatorial Mathematics IX. Proceedings of the Ninth Australian Conference on Combinatorial Mathematics, Brisbane, Australia, 24-28 August, 1981. Lecture Notes in Mathematics 952: 392-418. Springer-Verlag.

Street, A. P. and Macdonald, S. O. (1979). Balanced binary arrays I: The square grid. Combinatorial Mathematics VI. Proceedings of the Sixth Australian Conference (eds. A. F. Horadam and W. D. Wallis), Lecture Notes in Mathematics 748: 165-198. Springer-Verlag, Heidelberg.

Street, A. P. and Street, D. J. (1987). *Combinatorics of Experimental Design*. Clarendon press, London.

Street, A. P. and Wallis, W. D. (1977). *Combinatorial Theory: An Introduction*. Charles Babbage Research Centre, Manitoba, Canada.

Street, D. J., Eccleston, J. A. and Wilson, W. H. (1990). Tables of small optimal repeated measurements designs. *Austral. J. Statist.* 32(3): 345-359.

Street, D. J. and Wilson, W. H. (1985). Balanced designs for two-variety competition experiments. *Utilitas Mathematica* 28: 113-120.

The Dictionary of Forestry (1998). Society of American Foresters and CABI Publishing.

The Shorter Oxford English Dictionary on Historical Principles (1973). Third edition. Clarendon Press, Oxford.

Titsworth, R. C. (1964). Equivalence classes of periodic sequences. *Illinois Journal of Mathematics* 8: 266-270.

Toky, O. P., Kumar, P. and Khosla, P. K. (1989). Structure and function of traditional agroforestry systems in the western Himalaya. I. Biomass and productivity. *Agroforestry Systems* 9: 47-70.

Vanclay, J. K. (1986). Design for a gene recombination orchard. *Silvae Genetica* 35: 1-3.

Vanclay, J. K. (1991). Seed orchard designs by computer. *Silvae Genetica* 40: 89-91.

Veevers, A. (1982). Balanced designs for observing intra-variety nearest-neighbour interactions. *Euphytica* 31: 465-468.

Veevers, A. and Boffey, T. B. (1975). On the existence of levelled beehive designs. *Biometrics* 31: 963-967.

Veevers, A. and Boffey, T. B. (1979). Designs for balanced observation of plant competition. *Journal of Statistical Planning and Inference* 3: 325-331.

Veevers, A., Boffey, T. B. and Zafar-Yab, M. (1980). Competition experiments for two varieties. *Proceedings of the Tenth European Meeting of Statisticians*. Zeuven, Belgium.

Veevers, A. and Zafar-Yab, M. (1980). Balanced designs for two-component competition experiments on a square lattice. *Euphytica* 29: 459-464.

Wahua, T. A. T. and Miller, D. A. (1978). Relative yield totals and yield components of intercropped sorghum and soybeans. *Agronomy Journal* 70: 287-291.

Wallis, W. D. (1988). *Combinatorial Designs*. Marcel Dekker, Inc.

Watson, H. R. and Laquihon, W. A. (1987). Sloping agricultural land technology: an agroforestry model for soil conservation. In N. T. Vergara and N. D. Briones (eds.), *Agroforestry in the Humid Tropics: its protective and ameliorative roles to enhance productivity and sustainability*, Environment and Policy Institute, East-West Center, Honolulu,

Hawaii, USA; and Southeast Asian Regional Center for Graduate Study and Research in Agriculture, Los Baños, Laguna, Philippines: 209-226.

Weir, R. J. (1973). Realizing genetic gains through second-generation seed orchards. Proc. of the Twelfth Southern Forest Tree Improvement Conf., Baton Rouge, Louisiana, 12-13 June, 1973: 14-23.

Wheeler, N. C., Adams, W. T. and Hamrick, J. L. (1993). Pollen distribution in wind-pollinated seed orchards. In *Advances in Pollen Management*, Agriculture Handbook 698, Forest Service, United States Department of Agriculture: 25-31.

Willey, R. W. (1979a). Intercropping - its importance and research needs. Part 1. Competition and yield advantages. Field Crop Abstracts 32(1): 1-10.

Willey, R. W. (1979b). Intercropping - its importance and research needs. Part 2. Agronomy and research approaches. Field Crop Abstracts 32(2): 73-85.

Willey, R. W. and Heath, S. B. (1969). The quantitative relationships between plant population and crop yield. Advances in Agronomy 21: 281-321.

Willey, R. W. and Osiru, D. S. O. (1972). Studies on mixtures of maize and beans (*Phaseolus vulgaris*) with particular reference to plant population. J. agric. Sci., Camb. 79: 517-529.

Willey, R. W. and Rao, M. R. (1981). A systematic design to examine effects of plant population and spatial arrangement in intercropping, illustrated by an experiment on chickpea/safflower. Expl Agric. 17: 63-73.

Willey, R. W. and Reddy, M. S. (1981). A field technique for separating above- and below-ground interactions in intercropping: an experiment with pearl millet/groundnut. Expl Agric. 17: 257-264.

Williams, E. J. (1988). A survey of experimental design in Australia. *Austral. J. Statist.* 30(B): 54-76.

Williams, E. R. and Bailey, R. A. (1981). A note on designs for neighbour configurations. *Mathematical Biosciences* 56: 153-154.

de Wit, C. T. (1960). On competition. *Versl. Landbouwk. Onderzoek*. No. 66.8. (Institute for Biological and Chemical Research on Field Crops and Herbage, Wageningen, the Netherlands).

von Wuehlisch, G., Muhs, H.-J. and Geburek, T. (1990). Competitive behaviour of clones of *Picea abies* in monoclonal mosaics vs. intimate clonal mixtures. A pilot study. *Scand. J. For. Res.* 5: 397-401.

Yates, F. (1949). The design of rotation experiments. Proceedings of the 1st Commonwealth Conference on Tropical and Sub Tropical Soils, 1948. Commonwealth Bureau of Soil Sci. Tech. Comm. No. 46.

Zobel, B. J., Barber, J., Brown, C. L. and Perry, T. O. (1958). Seed orchards - their concept and management. *Journal of Forestry* 56: 815-825.

Zou, X. and Sanford Jr., R. L. (1990). Agroforestry systems in China: a survey and classification. *Agroforestry Systems* 11: 85-94.

## Appendix A. Counts of finite and periodic sequences.

Throughout this appendix, a shaded column indicates that the numbers in the column can be found in Sloane (1998).

Note that I have actually evaluated all formulae up to and including  $n = 31$ , but some tables have been truncated where they would require too small a font to fit comfortably on one page.

**Table A.1.** Total number of sequences with  $\leq q$  characters under the action of  $I_n \times I_q$ .

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	4	9	16	25	36
3	1	8	27	64	125	216
4	1	16	81	256	625	1296
5	1	32	243	1024	3125	7776
6	1	64	729	4096	15625	46656
7	1	128	2187	16384	78125	279936
8	1	256	6561	65536	390625	1679616
9	1	512	19683	262144	1953125	10077696
10	1	1024	59049	1048576	9765625	60466176
11	1	2048	177147	4194304	48828125	362797056
12	1	4096	531441	16777216	244140625	2176782336
13	1	8192	1594323	67108864	1220703125	13060694016
14	1	16384	4782969	268435456	6103515625	78364164096
15	1	32768	14348907	1073741824	30517578125	470184984576
16	1	65536	43046721	4294967296	152587890625	2821109907456
17	1	131072	129140163	17179869184	762939453125	16926659444736
18	1	262144	387420489	68719476736	3814697265625	101559956668416
19	1	524288	1162261467	274877906944	19073486328125	609359740010496
20	1	1048576	3486784401	1099511627776	95367431640625	3656158440062976
21	1	2097152	10460353203	4398046511104	476837158203125	21936950640377856
22	1	4194304	31381059609	17592186044416	2384185791015625	131621703842267136
23	1	8388608	94143178827	70368744177664	11920928955078125	789730223053602816
24	1	16777216	282429536481	281474976710656	59604644775390625	4738381338321616896

**Table A.2.** Total number of sequences with exactly  $q$  different characters under the action of  $I_n \times I_q$ .

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	2	0	0	0	0
3	1	6	6	0	0	0
4	1	14	36	24	0	0
5	1	30	150	240	120	0
6	1	62	540	1560	1800	720
7	1	126	1806	8400	16800	15120
8	1	254	5796	40824	126000	191520
9	1	510	18150	186480	834120	1905120
10	1	1022	55980	818520	5103000	16435440
11	1	2046	171006	3498000	29607600	129230640
12	1	4094	519156	14676024	165528000	953029440
13	1	8190	1569750	60780720	901020120	6711344640
14	1	16382	4733820	249401880	4809004200	45674188560
15	1	32766	14250606	1016542800	25292030400	302899156560
16	1	65534	42850116	4123173624	131542866000	1969147121760
17	1	131070	128746950	16664094960	678330198120	12604139926560
18	1	262142	386634060	67171367640	3474971465400	79694820748080
19	1	524286	1160688606	270232006800	17710714165200	499018753280880
20	1	1048574	3483638676	1085570781624	89904730860000	3100376804676480
21	1	2097150	10454061750	4356217681200	454951508208120	19141689213218880
22	1	4194302	31368476700	17466686971800	2296538629446600	117579844328562000
23	1	8388606	94118013006	69992221794000	11570026582092000	719258297748051600
24	1	16777214	282379204836	280345359228024	58200094019430000	4384969945980861600

**Table A.3.** Total number of primitive sequences with  $\leq q$  characters under the action of  $I_n \times I_q$ .

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	2	6	12	20	30
3	0	6	24	60	120	210
4	0	12	72	240	600	1260
5	0	30	240	1020	3120	7770
6	0	54	696	4020	15480	46410
7	0	126	2184	16380	78120	279930
8	0	240	6480	65280	390000	1678320
9	0	504	19656	262080	1953000	10077480
10	0	990	58800	1047540	9762480	60458370
11	0	2046	177144	4194300	48828120	362797050
12	0	4020	530640	16772880	244124400	2176734420
13	0	8190	1594320	67108860	1220703120	13060694010
14	0	16254	4780776	268419060	6103437480	78363884130
15	0	32730	14348640	1073740740	30517574880	470184976590
16	0	65280	43040160	4294901760	152587500000	2821108227840
17	0	131070	129140160	17179869180	762939453120	16926659444730
18	0	261576	387400104	68719210560	3814695297000	101559946544280
19	0	524286	1162261464	274877906940	19073486328120	609359740010490
20	0	1047540	3486725280	1099510578960	95367421874400	3656158379595540
21	0	2097018	10460350992	4398046494660	476837158124880	21936950640097710
22	0	4192254	31380882456	17592181850100	2384185742187480	131621703479470050
23	0	8388606	94143178824	70368744177660	11920928955078120	789730223053602810
24	0	16772880	282428998560	281474959868160	59604644530860000	4738381336143156240

**Table A.4.** Total number of primitive sequences with exactly  $q$  different characters under the action of  $I_n \times I_q$ .

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	2	0	0	0	0
3	0	6	6	0	0	0
4	0	12	36	24	0	0
5	0	30	150	240	120	0
6	0	54	534	1560	1800	720
7	0	126	1806	8400	16800	15120
8	0	240	5760	40800	126000	191520
9	0	504	18144	186480	834120	1905120
10	0	990	55830	818280	5102880	16435440
11	0	2046	171006	3498000	29607600	129230640
12	0	4020	518580	14674440	165526200	953028720
13	0	8190	1569750	60780720	901020120	6711344640
14	0	16254	4732014	249393480	4808987400	45674173440
15	0	32730	14250450	1016542560	25292030280	302899156560
16	0	65280	42844320	4123132800	131542740000	1969146930240
17	0	131070	128746950	16664094960	678330198120	12604139926560
18	0	261576	386615376	67171179600	3474970629480	79694818842240
19	0	524286	1160688606	270232006800	17710714165200	499018753280880
20	0	1047540	3483582660	1085569963080	89904725757000	3100376788241040
21	0	2097018	10454059938	4356217672800	454951508191320	19141689213203760
22	0	4192254	31368305694	17466683473800	2296538599839000	117579844199331360
23	0	8388606	94118013006	69992221794000	11570026582092000	719258297748051600
24	0	16772880	282378679920	280345344511200	58200093853776000	4384969945027640640

**Table A.5.** Total number of sequence structures with  $\leq q$  characters under the action of  $I_n \times S_q$ .

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	4	5	5	5	5
4	1	8	14	15	15	15
5	1	16	41	51	52	52
6	1	32	122	187	202	203
7	1	64	365	715	855	876
8	1	128	1094	2795	3845	4111
9	1	256	3281	11051	18002	20648
10	1	512	9842	43947	86472	109299
11	1	1024	29525	175275	422005	601492
12	1	2048	88574	700075	2079475	3403127
13	1	4096	265721	2798251	10306752	19628064
14	1	8192	797162	11188907	51263942	114700315
15	1	16384	2391485	44747435	255514355	676207628
16	1	32768	7174454	178973355	1275163905	4010090463
17	1	65536	21523361	715860651	6368612302	23874362200
18	1	131072	64570082	2863377067	31821472612	142508723651
19	1	262144	193710245	11453377195	159042661905	852124263684
20	1	524288	581130734	45813246635	795019337135	5101098232519
21	1	1048576	1743392201	183252462251	3974515030652	30560194493456
22	1	2097152	5230176602	733008800427	19870830712482	183176170057707
23	1	4194304	15690529805	2932033104555	99348921288655	1098318779272060
24	1	8388608	47071589414	11728128223915	496728911719165	6586964947803695
25	1	16777216	141214768241	46912504507051	2483597478617802	39510014478620232
26	1	33554432	423644304722	187650001250987	12417846161543552	237013033135668883
27	1	67108864	1270932914165	750599971449515	62088807129858605	1421890125134903156
28	1	134217728	3812798742494	3002399818689195	310442764649269995	8530588879837946391

**Table A.6.** Total number of sequence structures with exactly  $q$  different characters under the action of  $I_n \times S_q$ .

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	3	1	0	0	0
4	1	7	6	1	0	0
5	1	15	25	10	1	0
6	1	31	90	65	15	1
7	1	63	301	350	140	21
8	1	127	966	1701	1050	266
9	1	255	3025	7770	6951	2646
10	1	511	9330	34105	42525	22827
11	1	1023	28501	145750	246730	179487
12	1	2047	86526	611501	1379400	1323652
13	1	4095	261625	2532530	7508501	9321312
14	1	8191	788970	10391745	40075035	63436373
15	1	16383	2375101	42355950	210766920	420693273
16	1	32767	7141686	171798901	1096190550	2734926558
17	1	65535	21457825	694337290	5652751651	17505749898
18	1	131071	64439010	2798806985	28958095545	110687251039
19	1	262143	193448101	11259666950	147589284710	693081601779
20	1	524287	580606446	45232115901	749206090500	4306078895384
21	1	1048575	1742343625	181509070050	3791262568401	26585679462804
22	1	2097151	5228079450	727778623825	19137821912055	163305339345225
23	1	4194303	15686335501	2916342574750	96416888184100	998969857983405
24	1	8388607	47063200806	11681056634501	485000783495250	6090236036084530
25	1	16777215	141197991025	46771289738810	2436684974110751	37026417000002430
26	1	33554431	423610750290	187226356946265	12230196160292565	224595186974125331
27	1	67108863	1270865805301	749329038535350	61338207158409090	1359801318005044551
28	1	134217727	3812664524766	2998587019946701	307440364830580800	8220146115188676396

**Table A.7.** Total number of primitive sequence structures with  $\leq q$  characters under the action of  $I_n \times S_q$ .

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	0	3	4	4	4	4
4	0	6	12	13	13	13
5	0	15	40	50	51	51
6	0	27	116	181	196	197
7	0	63	364	714	854	875
8	0	120	1080	2780	3830	4096
9	0	252	3276	11046	17997	20643
10	0	495	9800	43895	86419	109246
11	0	1023	29524	175274	422004	601491
12	0	2010	88440	699875	2079260	3402911
13	0	4095	265720	2798250	10306751	19628063
14	0	8127	796796	11188191	51263086	114699438
15	0	16365	2391440	44747380	255514299	676207572
16	0	32640	7173360	178970560	1275160060	4010086352
17	0	65535	21523360	715860650	6368612301	23874362199
18	0	130788	64566684	2863365834	31821454413	142508702805
19	0	262143	193710244	11453377194	159042661904	852124263683
20	0	523770	581120880	45813202675	795019250650	5101098123207
21	0	1048509	1743391832	183252461532	3974515029793	30560194492576
22	0	2096127	5230147076	733008625151	19870830290476	183176169456214
23	0	4194303	15690529804	2932033104554	99348921288654	1098318779272059
24	0	8386440	47071499760	11728127521060	496728909635860	6586964944396472
25	0	16777200	141214768200	46912504507000	2483597478617750	39510014478620180
26	0	33550335	423644039000	187649998452735	12417846151236799	237013033116040818
27	0	67108608	1270932910884	750599971438464	62088807129840603	1421890125134882508
28	0	134209530	3812797945320	3002399807500275	310442764598006040	8530588879723246063

**Table A.8.** Total number of primitive sequence structures with exactly  $q$  different characters under the action of  $I_n \times S_q$ .

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	3	1	0	0	0
4	0	6	6	1	0	0
5	0	15	25	10	1	0
6	0	27	89	65	15	1
7	0	63	301	350	140	21
8	0	120	960	1700	1050	266
9	0	252	3024	7770	6951	2646
10	0	495	9305	34095	42524	22827
11	0	1023	28501	145750	246730	179487
12	0	2010	86430	611435	1379385	1323651
13	0	4095	261625	2532530	7508501	9321312
14	0	8127	788669	10391395	40074895	63436352
15	0	16365	2375075	42355940	210766919	420693273
16	0	32640	7140720	171797200	1096189500	2734926292
17	0	65535	21457825	694337290	5652751651	17505749898
18	0	130788	64435896	2798799150	28958088579	110687248392
19	0	262143	193448101	11259666950	147589284710	693081601779
20	0	523770	580597110	45232081795	749206047975	4306078872557
21	0	1048509	1742343323	181509069700	3791262568261	26585679462783
22	0	2096127	5228050949	727778478075	19137821665325	163305339165738
23	0	4194303	15686335501	2916342574750	96416888184100	998969857983405
24	0	8386440	47063113320	11681056021300	485000782114800	6090236034760612
25	0	16777200	141197991000	46771289738800	2436684974110750	37026417000002430
26	0	33550335	423610488665	187226354413735	12230196152784064	224595186964804019
27	0	67108608	1270865802276	749329038527580	61338207158402139	1359801318005041905
28	0	134209530	3812663735790	2998587009554955	307440364790505765	8220146115125240023

**Table A.9.** Total number of periodic sequences with  $\leq q$  characters under the action of  $C_n \times I_q$ .

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	11	24	45	76
4	1	6	24	70	165	336
5	1	8	51	208	629	1560
6	1	14	130	700	2635	7826
7	1	20	315	2344	11165	39996
8	1	36	834	8230	48915	210126
9	1	60	2195	29144	217045	1119796
10	1	108	5934	104968	976887	6047412
11	1	188	16107	381304	4438925	32981556
12	1	352	44368	1398500	20346485	181402676
13	1	632	122643	5162224	93900245	1004668776
14	1	1182	341802	19175140	435970995	5597460306
15	1	2192	956635	71582944	2034505661	31345666736
16	1	4116	2690844	268439590	9536767665	176319474366
17	1	7712	7596483	1010580544	44878791365	995685849696
18	1	14602	21524542	3817763740	211927736135	5642220380006
19	1	27596	61171659	14467258264	1003867701485	32071565263716
20	1	52488	174342216	54975633976	4768372070757	182807925027504
21	1	99880	498112275	209430787824	22706531350485	1044616697187576
22	1	190746	1426419858	799645010860	108372083629275	5982804736593846
23	1	364724	4093181691	3059510616424	518301258916445	34336096654504476
24	1	699252	11767920118	11728124734500	2483526875847735	197432555854243976
25	1	1342184	33891544419	45035996273872	11920928955078629	1137211521197189304
26	1	2581428	97764131646	173215372864600	57312158484825735	6560835699716880876
27	1	4971068	282429537947	667199944815064	275947429515842045	37907050706573681716
28	1	9587580	817028472960	2573485510942780	1330460821097243445	219319364805113579616

**Table A.10.** Total number of periodic sequences with exactly  $q$  different characters under the action of  $C_n \times I_q$ .

n	q					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	2	0	0	0
4	1	4	9	6	0	0
5	1	6	30	48	24	0
6	1	12	91	260	300	120
7	1	18	258	1200	2400	2160
8	1	34	729	5106	15750	23940
9	1	58	2018	20720	92680	211680
10	1	106	5613	81876	510312	1643544
11	1	186	15546	318000	2691600	11748240
12	1	350	43315	1223136	13794150	79419180
13	1	630	120750	4675440	69309240	516257280
14	1	1180	338259	17815020	343501500	3262443120
15	1	2190	950062	67769552	1686135376	20193277104
16	1	4114	2678499	257700906	8221437000	123071707080
17	1	7710	7573350	980240880	39901776360	741419995680
18	1	14600	21480739	3731753180	193054016840	4427490147480
19	1	27594	61088874	14222737200	932142850800	26264144909520
20	1	52486	174184755	54278580036	4495236798162	155018841055596
21	1	99878	497812638	207438938000	21664357535320	911509010154720
22	1	190744	1425847623	793940475900	104388120866100	5344538384445120
23	1	364722	4092087522	3043140078000	503044634004000	31272099902089200
24	1	699250	11765822365	11681057249536	2425003924383900	182707081122261480
25	1	1342182	33887517870	44900438149296	11696087875731624	1066360809600069984
26	1	2581426	97756387365	172824331826580	56447059236004920	6219559024156983960
27	1	4971066	282414624746	666070256489680	272614254037435520	36261368480134662480
28	1	9587578	816999710223	2570217454576416	1317601563731383350	211375185820768615140

**Table A.11.** Total number of primitive periodic sequences with  $\leq q$  characters under the action of  $C_n \times I_q$ .

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	1	3	6	10	15
3	0	2	8	20	40	70
4	0	3	18	60	150	315
5	0	6	48	204	624	1554
6	0	9	116	670	2580	7735
7	0	18	312	2340	11160	39990
8	0	30	810	8160	48750	209790
9	0	56	2184	29120	217000	1119720
10	0	99	5880	104754	976248	6045837
11	0	186	16104	381300	4438920	32981550
12	0	335	44220	1397740	20343700	181394535
13	0	630	122640	5162220	93900240	1004668770
14	0	1161	341484	19172790	435959820	5597420295
15	0	2182	956576	71582716	2034504992	31345665106
16	0	4080	2690010	268431360	9536718750	176319264240
17	0	7710	7596480	1010580540	44878791360	995685849690
18	0	14532	21522228	3817733920	211927516500	5642219252460
19	0	27594	61171656	14467258260	1003867701480	32071565263710
20	0	52377	174336264	54975528948	4768371093720	182807918979777
21	0	99858	498111952	209430785460	22706531339280	1044616697147510
22	0	190557	1426403748	799644629550	108372079190340	5982804703612275
23	0	364722	4093181688	3059510616420	518301258916440	34336096654504470
24	0	698870	11767874940	11728123327840	2483526855452500	197432555672631510
25	0	1342176	33891544368	45035996273664	11920928955078000	1137211521197187744
26	0	2580795	97764009000	173215367702370	57312158390925480	6560835698712212085
27	0	4971008	282429535752	667199944785920	275947429515625000	37907050706572561920
28	0	9586395	817028131140	2573485491767580	1330460820661272300	219319364799516118995

**Table A.12.** Total number of primitive periodic sequences with exactly  $q$  different characters under the action of  $C_n \times I_q$ .

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	2	0	0	0
4	0	3	9	6	0	0
5	0	6	30	48	24	0
6	0	9	89	260	300	120
7	0	18	258	1200	2400	2160
8	0	30	720	5100	15750	23940
9	0	56	2016	20720	92680	211680
10	0	99	5583	81828	510288	1643544
11	0	186	15546	318000	2691600	11748240
12	0	335	43215	1222870	13793850	79419060
13	0	630	120750	4675440	69309240	516257280
14	0	1161	338001	17813820	343499100	3262440960
15	0	2182	950030	67769504	1686135352	20193277104
16	0	4080	2677770	257695800	8221421250	123071683140
17	0	7710	7573350	980240880	39901776360	741419995680
18	0	14532	21478632	3731732200	193053923860	4427489935680
19	0	27594	61088874	14222737200	932142850800	26264144909520
20	0	52377	174179133	54278498154	4495236287850	155018839412052
21	0	99858	497812378	207438936800	21664357532920	911509010152560
22	0	190557	1425832077	793940157900	104388118174500	5344538372696880
23	0	364722	4092087522	3043140078000	503044634004000	31272099902089200
24	0	698870	11765778330	11681056021300	2425003910574000	182707081042818360
25	0	1342176	33887517840	44900438149248	11696087875731600	1066360809600069984
26	0	2580795	97756266615	172824327151140	56447059166695680	6219559023640726680
27	0	4971008	282414622728	666070256468960	272614254037342840	36261368480134450800
28	0	9586395	816999371955	2570217436761390	1317601563387881850	211375185817506172020

**Table A.13.** Total number of periodic sequence structures with  $\leq q$  characters under the action of  $C_n \times S_q$  where  $C_n$  is the group of cyclic shifts.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	4	6	7	7	7
5	1	4	9	11	12	12
6	1	8	26	39	42	43
7	1	10	53	103	123	126
8	1	20	146	367	503	539
9	1	30	369	1235	2008	2304
10	1	56	1002	4439	8720	11023
11	1	94	2685	15935	38365	54682
12	1	180	7434	58509	173609	284071
13	1	316	20441	215251	792828	1509852
14	1	596	57046	799697	3662924	8195029
15	1	1096	159451	2983217	17034381	45080666
16	1	2068	448686	11187567	79703081	250641895
17	1	3856	1266081	42109451	374624254	1404374248
18	1	7316	3588002	159082753	1767883444	7917211349
19	1	13798	10195277	602809327	8370666417	44848645458
20	1	26272	29058526	2290684251	39751072847	255055231763
21	1	49940	83018783	8726308317	189262621864	1455247360128
22	1	95420	237740670	33318661277	903220058756	8326191290585
23	1	182362	682196949	127479700199	4319518316899	47752990403134
24	1	349716	1961331314	488672302909	20697040198889	274456882422179
25	1	671092	5648590737	1876500180291	99343899144822	1580400579145068
26	1	1290872	16294052602	7217308815887	477609477924308	9115885942871311
27	1	2485534	47071590147	27799998949873	2299585449279713	52662597227306274
28	1	4794088	136171497650	107228568948547	11087241641849179	304663888861106659
29	1	9256396	394427456121	414124108294451	53524483114254820	1764845760906069964
30	1	17896832	1143839943618	1601279891396753	258701287317263442	10235704713649961711

**Table A.14.** Total number of periodic sequence structures with exactly  $q$  different characters under the action of  $C_n \times S_q$ , where  $C_n$  is the group of cyclic shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	3	2	1	0	0
5	1	3	5	2	1	0
6	1	7	18	13	3	1
7	1	9	43	50	20	3
8	1	19	126	221	136	36
9	1	29	339	866	773	296
10	1	55	946	3437	4281	2303
11	1	93	2591	13250	22430	16317
12	1	179	7254	51075	115100	110462
13	1	315	20125	194810	577577	717024
14	1	595	56450	742651	2863227	4532105
15	1	1095	158355	2823766	14051164	28046285
16	1	2067	446618	10738881	68515514	170938814
17	1	3855	1262225	40843370	332514803	1029749994
18	1	7315	3580686	155494751	1608800691	6149327905
19	1	13797	10181479	592614050	7767857090	36477979041
20	1	26271	29032254	2261625725	37460388596	215304158916
21	1	49939	82968843	8643289534	180536313547	1265984738264
22	1	95419	237645250	33080920607	869901397479	7422971231829
23	1	182361	682014587	126797503250	4192038616700	43433472086235
24	1	349715	1960981598	486710971595	20208367895980	253759842223290
25	1	671091	5647919645	1870851589554	97467398964531	1481056680000246
26	1	1290871	16292761730	7201014763285	470392169108421	8638276464947003
27	1	2485533	47069104613	27752927359726	2271785450329840	50363011778026561
28	1	4794087	136166703562	107092397450897	10980013072900632	293576647219257480
29	1	9256395	394418199725	413729680838330	53110359005960369	1711321277791815144
30	1	17896831	1143822046786	1600136051453135	257100007425866689	9977003426332698269

**Table A.15.** Total number of primitive periodic sequence structures with  $\leq q$  characters under the action of  $C_n \times S_q$ , where  $C_n$  is the group of cyclic shifts.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	0	1	2	2	2	2
4	0	2	4	5	5	5
5	0	3	8	10	11	11
6	0	5	22	35	38	39
7	0	9	52	102	122	125
8	0	16	140	360	496	532
9	0	28	366	1232	2005	2301
10	0	51	992	4427	8707	11010
11	0	93	2684	15934	38364	54681
12	0	170	7404	58465	173562	284023
13	0	315	20440	215250	792827	1509851
14	0	585	56992	799593	3662800	8194902
15	0	1091	159440	2983204	17034367	45080652
16	0	2048	448540	11187200	79702578	250641356
17	0	3855	1266080	42109450	374624253	1404374247
18	0	7280	3587610	159081482	1767881397	7917209005
19	0	13797	10195276	602809326	8370666416	4484645457
20	0	26214	29057520	2290679807	39751064122	255055220735
21	0	49929	83018728	8726308212	189262621739	1455247360000
22	0	95325	237737984	33318645341	903220020390	8326191235902
23	0	182361	682196948	127479700198	4319518316898	47752990403133
24	0	349520	1961323740	488672244040	2069704024784	274456882137576
25	0	671088	5648590728	1876500180280	99343899144810	1580400579145056
26	0	1290555	16294032160	7217308600635	477609477131479	9115885941361458
27	0	2485504	47071589778	27799998948638	2299585449277705	52662597227303970
28	0	4793490	136171440600	107228568148845	11087241638186250	304663888852911625
29	0	9256395	394427456120	414124108294450	53524483114254819	1764845760906069963

**Table A.16.** Total number of primitive periodic sequence structures with exactly  $q$  different characters under the action of  $C_n \times S_q$ , where  $C_n$  is the group of cyclic shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	1	1	0	0	0
4	0	2	2	1	0	0
5	0	3	5	2	1	0
6	0	5	17	13	3	1
7	0	9	43	50	20	3
8	0	16	124	220	136	36
9	0	28	338	866	773	296
10	0	51	941	3435	4280	2303
11	0	93	2591	13250	22430	16317
12	0	170	7234	51061	115097	110461
13	0	315	20125	194810	577577	717024
14	0	585	56407	742601	2863207	4532102
15	0	1091	158349	2823764	14051163	28046285
16	0	2048	446492	10738660	68515378	170938778
17	0	3855	1262225	40843370	332514803	1029749994
18	0	7280	3580330	155493872	1608799915	6149327608
19	0	13797	10181479	592614050	7767857090	36477979041
20	0	26214	29031306	2261622287	37460384315	215304156613
21	0	49929	82968799	8643289484	180536313527	1265984738261
22	0	95325	237642659	33080907357	869901375049	7422971215512
23	0	182361	682014587	126797503250	4192038616700	43433472086235
24	0	349520	1960974220	486710920300	20208367780744	253759842112792
25	0	671088	5647919640	1870851589552	97467398964530	1481056680000246
26	0	1290555	16292741605	7201014568475	470392168530844	8638276464229979
27	0	2485504	47069104274	27752927358860	2271785450329067	50363011778026265
28	0	4793490	136166647110	107092396708245	10980013070037405	293576647214725375
29	0	9256395	394418199725	413729680838330	53110359005960369	1711321277791815144
30	0	17895679	1143821887473	1600136048625921	257100007411811242	9977003426304649680

**Table A.17.** Total number of finite sequences with  $\leq q$  characters under the action of  $R_n \times I_q$ , where  $R_n$  is the group of reversals.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	6	18	40	75	126
4	1	10	45	136	325	666
5	1	20	135	544	1625	3996
6	1	36	378	2080	7875	23436
7	1	72	1134	8320	39375	140616
8	1	136	3321	32896	195625	840456
9	1	272	9963	131584	978125	5042736
10	1	528	29646	524800	4884375	30236976
11	1	1056	88938	2099200	24421875	181421856
12	1	2080	266085	8390656	122078125	1088414496
13	1	4160	798255	33562624	610390625	6530486976
14	1	8256	2392578	134225920	3051796875	39182222016
15	1	16512	7177734	536903680	15258984375	235093332096
16	1	32896	21526641	2147516416	76294140625	1410555793536
17	1	65792	64579923	8590065664	381470703125	8463334761216
18	1	131328	193720086	34359869440	1907349609375	50779983373056
19	1	262656	581160258	137439477760	9536748046875	304679900238336
20	1	524800	1743421725	549756338176	47683720703125	1828079250264576
21	1	1049600	5230265175	2199025352704	238418603515625	10968475501587456
22	1	2098176	15690618378	8796095119360	1192092919921875	65810852102532096
23	1	4196352	47071855134	35184380477440	5960464599609375	394865112615192576
24	1	8390656	141215033961	140737496743936	29802322509765625	2369190670249199616
25	1	16781312	423645101883	562949986975744	14901161254882125	14215144021495197696
26	1	33558528	1270933711326	2251799847239680	745058060302734375	85290864096319451136
27	1	67117056	3812801133978	9007199388958720	3725290301513671875	511745184577916706816

**Table A.18.** Total number of finite sequences with exactly  $q$  different characters under the action of  $R_n \times I_q$ , where  $R_n$  is the group of reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	4	3	0	0	0
4	1	8	18	12	0	0
5	1	18	78	120	60	0
6	1	34	273	780	900	360
7	1	70	921	4212	8400	7560
8	1	134	2916	20424	63000	95760
9	1	270	9150	93360	417120	952560
10	1	526	28065	409380	2551560	8217720
11	1	1054	85773	1749780	14804700	64615680
12	1	2078	259848	7338792	82764900	476515080
13	1	4158	785778	30394560	450518460	3355679880
14	1	8254	2367813	124705140	2404510500	22837101840
15	1	16510	7128201	508291812	12646078200	151449674040
16	1	32894	21427956	2061607224	65771496000	984573656640
17	1	65790	64382550	8332140720	339165516120	6302070915840
18	1	131326	193326105	33585777060	1737486149760	39847411326600
19	1	262654	580372293	135116412660	8855359634100	249509384858160
20	1	524798	1741847328	542785800072	44952367981500	1550188410555960
21	1	1049598	5227116378	2178110589600	227475768907860	9570844671224760
22	1	2098174	15684323853	8733345234900	1148269329527100	5878992228896320
23	1	4196350	47059266081	34996118235012	5785013373810000	359629149350540520
24	1	8390654	141189861996	140172686952024	29100047092479000	2192484973466945520
25	1	16781310	423594757950	561255507256080	146201098897155120	13329510123356547120
26	1	33558526	1270833035745	2246716313745540	733811770068063960	80854267314040791480
27	1	67117054	3812599782813	8991948587125140	3680292431909047500	489528474504653132640

**Table A.19.** Total number of primitive finite sequences with  $\leq q$  characters under the action of  $R_n \times I_q$ , where  $R_n$  is the group of reversals.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	1	3	6	10	15
3	0	4	15	36	70	120
4	0	7	39	126	310	645
5	0	18	132	540	1620	3990
6	0	29	357	2034	7790	23295
7	0	70	1131	8316	39370	140610
8	0	126	3276	32760	195300	839790
9	0	266	9945	131544	978050	5042610
10	0	507	29508	524250	4882740	30232965
11	0	1054	88935	2099196	24421870	181421850
12	0	2037	265668	8388450	122069940	1088390415
13	0	4158	798252	33562620	610390620	6530486970
14	0	8183	2391441	134217594	3051757490	39182081385
15	0	16488	7177584	536903100	15258982680	235093327980
16	0	32760	21523320	2147483520	76293945000	1410554953080
17	0	65790	64579920	8590065660	381470703120	8463334761210
18	0	131026	193709763	34359735816	1907348623450	50779978307010
19	0	262654	581160255	137439477756	9536748046870	304679900238330
20	0	524265	1743392040	549755813250	47683715818440	1828079220026955
21	0	1049524	5230264026	2199025344348	238418603476180	10968475501446720
22	0	2097119	15690529437	8796093020154	1192092895499990	65810851921110225
23	0	4196350	47071855131	35184380477436	5960464599609370	394865112615192570
24	0	8388450	141214764600	140737488320520	29802322387492200	2369190669159945330
25	0	16781292	423645101748	562949986975200	149011612548826500	14215144021495193700
26	0	33554367	1270932913068	2251799813677050	745058059692343740	85290864089788964145
27	0	67116784	3812801124015	9007199388827136	3725290301512693750	511745184577911664080

**Table A.20.** Total number of primitive finite sequences with exactly  $q$  different characters under the action of  $R_n \times I_q$ , where  $R_n$  is the group of reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	4	3	0	0	0
4	0	7	18	12	0	0
5	0	18	78	120	60	0
6	0	29	270	780	900	360
7	0	70	921	4212	8400	7560
8	0	126	2898	20412	63000	95760
9	0	266	9147	93360	417120	952560
10	0	507	27987	409260	2551500	8217720
11	0	1054	85773	1749780	14804700	64615680
12	0	2037	259557	7338000	82764000	476514720
13	0	4158	785778	30394560	450518460	3355679880
14	0	8183	2366892	124700928	2404502100	22837094280
15	0	16488	7128120	508291692	12646078140	151449674040
16	0	32760	21425040	2061586800	65771433000	984573560880
17	0	65790	64382550	8332140720	339165516120	6302070915840
18	0	131026	193316685	33585682920	1737485731740	39847410373680
19	0	262654	580372293	135116412660	8855359634100	249509384858160
20	0	524265	1741819245	542785390680	44952365429940	1550188402338240
21	0	1049524	5227115454	2178110585388	227475768899460	9570844671217200
22	0	2097119	15684238080	8733343485120	1148269314722400	58789922164280640
23	0	4196350	47059266081	34996118235012	5785013373810000	359629149350540520
24	0	8388450	141189599250	140172679592820	29100047009651100	2192484972990334680
25	0	16781292	423594757872	561255507255960	146201098897155060	13329510123356547120
26	0	33554367	1270832249967	2246716283350980	733811769617545500	80854267310685111600
27	0	67116784	3812599773663	8991948587031780	3680292431908630380	489528474504652180080

**Table A.21.** Total number of finite sequence structures with  $\leq q$  characters under the action of  $R_n \times S_q$ , where  $R_n$  is the group of reversals.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	3	4	4	4	4
4	1	6	10	11	11	11
5	1	10	25	31	32	32
6	1	20	70	107	116	117
7	1	36	196	379	455	467
8	1	72	574	1451	1993	2135
9	1	136	1681	5611	9134	10480
10	1	272	5002	22187	43580	55091
11	1	528	14884	87979	211659	301633
12	1	1056	44530	350891	1041441	1704115
13	1	2080	133225	1400491	5156642	9819216
14	1	4160	399310	5597867	25640456	57365191
15	1	8256	1196836	22379179	127773475	338134521
16	1	16512	3589414	89500331	637624313	2005134639
17	1	32896	10764961	357952171	3184387574	11937364184
18	1	65792	32291602	1431743147	15910947980	71254895955
19	1	131328	96864964	5726775979	79521737939	426063226937
20	1	262656	290585050	22906841771	397510726681	2550552314219
21	1	524800	871725625	91626580651	1987259550002	15280103807200
22	1	1049600	2615147350	366505274027	9935420646296	91588104196415
23	1	2098176	7845353476	1466017950379	49674470817195	549159428968825
24	1	4196352	23535971854	5864067607211	248364482308833	3293482588847143
25	1	8390656	70607649841	23456257845931	1241798790172214	19755007475217288
26	1	16781312	211822683802	93825014606507	6208923213015980	118506517257327019
27	1	33558528	635467254244	375300008094379	31044403819243819	710945063982626841

**Table A.22.** Total number of finite sequence structures with exactly  $q$  different characters under the action of  $R_n \times S_q$ , where  $R_n$  is the group of reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	1	0	0	0
4	1	5	4	1	0	0
5	1	9	15	6	1	0
6	1	19	50	37	9	1
7	1	35	160	183	76	12
8	1	71	502	877	542	142
9	1	135	1545	3930	3523	1346
10	1	271	4730	17185	21393	11511
11	1	527	14356	73095	123680	89974
12	1	1055	43474	306361	690550	662674
13	1	2079	131145	1267266	3756151	4662574
14	1	4159	395150	5198557	20042589	31724735
15	1	8255	1188580	21182343	105394296	210361046
16	1	16511	3572902	85910917	548123982	1367510326
17	1	32895	10732065	347187210	2826435403	8752976610
18	1	65791	32225810	1399451545	14479204833	55343947975
19	1	131327	96733636	5629911015	73794961960	346541488998
20	1	262655	290322394	22616256721	374603884910	2153041587538
21	1	524799	871200825	90754855026	1895632969351	13292844257198
22	1	1049599	2614097750	363890126677	9568915372269	81652683550119
23	1	2098175	7843255300	1458172596903	48208452866816	499484958151630
24	1	4196351	23531775502	5840531635357	242500414701622	3045118106538310
25	1	8390655	70599259185	23385650196090	1218342532326283	18513208685045074
26	1	16781311	211805902490	93613191922705	6115098198409473	112297594044311039
27	1	33558527	635433695716	374664540840135	30669103811149440	679900660163383022

**Table A.23.** Total number of primitive finite sequence structures with  $\leq q$  characters under the action of  $R_n \times S_q$ , where  $R_n$  is the group of reversals.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	0	2	3	3	3	3
4	0	4	8	9	9	9
5	0	9	24	30	31	31
6	0	16	65	102	111	112
7	0	35	195	378	454	466
8	0	66	564	1440	1982	2124
9	0	133	1677	5607	9130	10476
10	0	261	4976	22155	43547	55058
11	0	527	14883	87978	211658	301632
12	0	1032	44452	350775	1041316	1703989
13	0	2079	133224	1400490	5156641	9819215
14	0	4123	399113	5597487	25640000	57364723
15	0	8244	1196808	22379145	127773440	338134486
16	0	16440	3588840	89498880	637622320	2005132504
17	0	32895	10764960	357952170	3184387573	11937364183
18	0	65639	32289855	1431737433	15910938734	71254885362
19	0	131327	96864963	5726775978	79521737938	426063226936
20	0	262380	290580040	22906819575	397510683092	2550552259119
21	0	524762	871725426	91626580269	1987259549544	15280103806730
22	0	1049071	2615132465	366505186047	9935420434636	91588103894781
23	0	2098175	7845353475	1466017950378	49674470817194	549159428968824
24	0	4195230	23535926760	5864067254880	248364481265410	3293482587140904
25	0	8390646	70607649816	23456257845900	1241798790172182	19755007475217256
26	0	16779231	211822550576	93825013206015	6208923207859337	118506517247507802
27	0	33558392	635467252563	375300008088768	31044403819234685	710945063982616361

**Table A.24.** Total number of primitive finite sequence structures with exactly  $q$  different characters under the action of  $R_n \times S_q$ , where  $R_n$  is the group of reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	1	0	0	0
4	0	4	4	1	0	0
5	0	9	15	6	1	0
6	0	16	49	37	9	1
7	0	35	160	183	76	12
8	0	66	498	876	542	142
9	0	133	1544	3930	3523	1346
10	0	261	4715	17179	21392	11511
11	0	527	14356	73095	123680	89974
12	0	1032	43420	306323	690541	662673
13	0	2079	131145	1267266	3756151	4662574
14	0	4123	394990	5198374	20042513	31724723
15	0	8244	1188564	21182337	105394295	210361046
16	0	16440	3572400	85910040	548123440	1367510184
17	0	32895	10732065	347187210	2826435403	8752976610
18	0	65639	32224216	1399447578	14479201301	55343946628
19	0	131327	96733636	5629911015	73794961960	346541488998
20	0	262380	290317660	22616239535	374603863517	2153041576027
21	0	524762	871200664	90754854843	1895632969275	13292844257186
22	0	1049071	2614083394	363890053582	9568915248589	81652683460145
23	0	2098175	7843255300	1458172596903	48208452866816	499484958151630
24	0	4195230	23531731530	5840531328120	242500414010530	3045118105875494
25	0	8390646	70599259170	23385650196084	1218342532326282	18513208685045074
26	0	16779231	211805771345	93613190655439	6115098194653322	112297594039648465
27	0	33558392	635433694171	374664540836205	30669103811145917	679900660163381676

**Table A.25.** Total number of inequivalent sequences with  $\leq q$  characters under the action of  $D_n \times I_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	10	20	35	56
4	1	6	21	55	120	231
5	1	8	39	136	377	888
6	1	13	92	430	1505	4291
7	1	18	198	1300	5895	20646
8	1	30	498	4435	25395	107331
9	1	46	1219	15084	110085	563786
10	1	78	3210	53764	493131	3037314
11	1	126	8418	192700	2227275	16514106
12	1	224	22913	704370	10196680	90782986
13	1	380	62415	2589304	46989185	502474356
14	1	687	173088	9608050	218102685	2799220041
15	1	1224	481598	35824240	1017448143	15673673176
16	1	2250	1351983	134301715	4768969770	88162676511
17	1	4112	3808083	505421344	22440372245	497847963696
18	1	7685	10781954	1909209550	105966797755	2821127825971
19	1	14310	30615354	7234153420	501938733555	16035812864946
20	1	27012	87230157	27489127708	2384200683816	91404068329560
21	1	50964	249144711	104717491064	11353290089305	522308529992316
22	1	96909	713387076	399827748310	54186115056825	2991403003191771
23	1	184410	2046856566	1529763696820	259150751528535	17168049415643406
24	1	352698	5884491500	5864083338770	1241763804134805	98716281736491076
25	1	675188	16946569371	22518031691368	5960465087890877	568605767128941660
26	1	1296858	48883660146	86607770318380	28656081073467555	3280417872714654966
27	1	2493726	141217160458	333600106625260	137973717809678835	18953525392468922906
28	1	4806078	408519019449	1286743091015710	665230419703895160	109659682539694076976
29	1	9272780	1183289542815	4969489780865944	3211457169105266705	635269884490073405796

**Table A.26.** Total number of inequivalent sequences with exactly  $q$  different characters under the action of  $D_n \times I_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	1	0	0	0
4	1	4	6	3	0	0
5	1	6	18	24	12	0
6	1	11	56	136	150	60
7	1	16	147	612	1200	1080
8	1	28	411	2619	7905	11970
9	1	44	1084	10480	46400	105840
10	1	76	2979	41388	255636	821952
11	1	124	8043	159780	1346700	5874480
12	1	222	22244	614058	6901725	39713550
13	1	378	61278	2341920	34663020	258136200
14	1	685	171030	8919816	171786450	1631273220
15	1	1222	477929	33905188	843130688	10096734312
16	1	2248	1345236	128907279	4110958530	61536377700
17	1	4110	3795750	490213680	19951305240	370710950400
18	1	7683	10758902	1866127840	96528492700	2213749658880
19	1	14308	30572427	7111777860	466073976900	13132080672480
20	1	27010	87149124	27140369148	2247627076731	77509456944318
21	1	50962	248991822	103721218000	10832193571460	455754569692680
22	1	96907	713096352	396974781456	52194109216950	2672269462787580
23	1	184408	2046303339	1521577377012	251522399766000	15636050427559320
24	1	352696	5883433409	5840547488954	1212502228828980	91353542477224260
25	1	675186	16944543810	22450249465008	5848044388375872	533180408155707312
26	1	1296856	48879769575	86412243458940	28223531045508540	3109779525174875280
27	1	2493724	141209679283	333035252945780	136307129423219860	18130684262904425520
28	1	4806076	408504601218	1285109043774378	658800789390950325	105687592997527643850
29	1	9272778	1183261724478	4964756678331360	3186621553003637340	616075660156503030120

**Table A.27.** Total number of primitive inequivalent sequences with  $\leq q$  characters under the action of  $D_n \times I_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	1	3	6	10	15
3	0	2	7	16	30	50
4	0	3	15	45	105	210
5	0	6	36	132	372	882
6	0	8	79	404	1460	4220
7	0	16	195	1296	5890	20640
8	0	24	477	4380	25275	107100
9	0	42	1209	15064	110050	563730
10	0	69	3168	53622	492744	3036411
11	0	124	8415	192696	2227270	16514100
12	0	208	22806	703895	10195070	90778485
13	0	378	62412	2589300	46989180	502474350
14	0	668	172887	9606744	218096780	2799199380
15	0	1214	481552	35824088	1017447736	15673672238
16	0	2220	1351485	134297280	4768944375	88162569180
17	0	4110	3808080	505421340	22440372240	497847963690
18	0	7630	10780653	1909194056	105966686200	2821127257950
19	0	14308	30615351	7234153416	501938733550	16035812864940
20	0	26931	87226932	27489073899	2384200190580	91404065292036
21	0	50944	249144506	104717489748	11353290083380	522308529971620
22	0	96782	713378655	399827555604	54186112829540	2991402986677650
23	0	184408	2046856563	1529763696816	259150751528530	17168049415643400
24	0	352450	5884468110	5864082630020	1241763793912850	98716281645600990
25	0	675180	16946569332	22518031691232	5960465087890500	568605767128940772
26	0	1296477	48883597728	86607767729070	28656081026478360	3280417872212180595
27	0	2493680	141217159239	333600106610176	137973717809568750	18953525392468359120
28	0	4805388	408518846346	1286743081407615	665230419485792370	109659682536894856725
29	0	9272778	1183289542812	4969489780865940	3211457169105266700	635269884490073405790

**Table A.28.** Total number of primitive inequivalent sequences with exactly  $q$  different characters under the action of  $D_n \times I_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	1	0	0	0
4	0	3	6	3	0	0
5	0	6	18	24	12	0
6	0	8	55	136	150	60
7	0	16	147	612	1200	1080
8	0	24	405	2616	7905	11970
9	0	42	1083	10480	46400	105840
10	0	69	2961	41364	255624	821952
11	0	124	8043	159780	1346700	5874480
12	0	208	22182	613919	6901575	39713490
13	0	378	61278	2341920	34663020	258136200
14	0	668	170883	8919204	171785250	1631272140
15	0	1214	477910	33905164	843130676	10096734312
16	0	2220	1344825	128904660	4110950625	61536365730
17	0	4110	3795750	490213680	19951305240	370710950400
18	0	7630	10757763	1866117224	96528446150	2213749552980
19	0	14308	30572427	7111777860	466073976900	13132080672480
20	0	26931	87146139	27140327757	2247626821095	77509456122366
21	0	50944	248991674	103721217388	10832193570260	455754569691600
22	0	96782	713088309	396974621676	52194107870250	2672269456913100
23	0	184408	2046303339	1521577377012	251522399766000	15636050427559320
24	0	352450	5883410760	5840546872280	1212502221919350	91353542437498740
25	0	675180	16944543792	22450249464984	5848044388375860	533180408155707312
26	0	1296477	48879708297	86412241117020	28223531010845520	3109779524916739080
27	0	2493680	141209678199	333035252935300	136307129423173460	18130684262904319680
28	0	4805388	408504430182	1285109034854559	658800789219163875	105687592995896370630
29	0	9272778	1183261724478	4964756678331360	3186621553003637340	616075660156503030120

**Table A.29.** Total number of inequivalent sequence structures with  $\leq q$  characters under the action of  $D_n \times S_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	4	6	7	7	7
5	1	4	9	11	12	12
6	1	8	22	33	36	37
7	1	9	40	73	89	92
8	1	18	100	237	322	349
9	1	23	225	703	1137	1308
10	1	44	582	2433	4704	5953
11	1	63	1464	8309	19839	28228
12	1	122	3960	30108	88508	144587
13	1	190	10585	108991	399680	760110
14	1	362	29252	403262	1839947	4112548
15	1	612	80819	1497070	8533488	22571040
16	1	1162	226530	5607437	39893901	125410355
17	1	2056	636321	21076571	187393550	702370208
18	1	3914	1800562	79595990	884153396	3959139804
19	1	7155	5107480	301492045	4185740195	22425417824
20	1	13648	14548946	1145560579	19876594537	127530813841
21	1	25482	41538916	4363503684	94633345608	727630240536
22	1	48734	118929384	16660204452	451615319433	4163114812854
23	1	92205	341187048	63741248201	2159769331317	23876534534362
24	1	176906	980842804	244339646708	10348546548695	137228556156385
25	1	337594	2824561089	938255682551	49672000435724	790200525479706
26	1	649532	8147557742	3608668388957	238804871206358	4557943660928233
27	1	1246863	23536592235	13900021844558	1149792978954373	26331300028828400
28	1	2405236	68087343148	53614340398327	5543621482141513	152331948566973257
29	1	4636390	197216119545	207062143625711	26762242828695896	882422888943285302
30	1	8964800	571924754778	800640169394590	129350646964708146	5117852381641897905
31	1	17334801	1660419530056	3099251723043169	625889663596316105	29715786751698562814

**Table A.30.** Total number of inequivalent sequence structures with exactly  $q$  different characters under the action of  $D_n \times S_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	3	2	1	0	0
5	1	3	5	2	1	0
6	1	7	14	11	3	1
7	1	8	31	33	16	3
8	1	17	82	137	85	27
9	1	22	202	478	434	171
10	1	43	538	1851	2271	1249
11	1	62	1401	6845	11530	8389
12	1	121	3838	26148	58400	56079
13	1	189	10395	98406	290689	360430
14	1	361	28890	374010	1436685	2272601
15	1	611	80207	1416251	7036418	14037552
16	1	1161	225368	5380907	34286464	85516454
17	1	2055	634265	20440250	166316979	514976658
18	1	3913	1796648	77795428	804557406	3074986408
19	1	7154	5100325	296384565	3884248150	18239677629
20	1	13647	14535298	1131011633	18731033958	107654219304
21	1	25481	41513434	4321964768	90269841924	632996894928
22	1	48733	118880650	16541275068	434955114981	3711499493421
23	1	92204	341094843	63400061153	2096028083116	21716765203045
24	1	176905	980665898	243358803904	10104206901987	126880009607690
25	1	337593	2824223495	935431121462	48733744753173	740528525043982
26	1	649531	8146908210	3600520831215	235196202817401	4319138789721875
27	1	1246862	23535345372	13876485252323	1135892957109815	25181507049874027
28	1	2405235	68084937912	53546253055179	5490007141743186	146788327084831744
29	1	4636389	197211483155	206864927506166	26555180685070185	855660646114589406
30	1	8964799	571915789978	800068244639812	128550006795313556	4988501734677189759
31	1	17334800	1660402195255	3097591303513113	622790411873272936	29089897088102246709

**Table A.31.** Total number of primitive inequivalent sequence structures with  $\leq q$  characters under the action of  $D_n \times S_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	0	1	2	2	2	2
4	0	2	4	5	5	5
5	0	3	8	10	11	11
6	0	5	18	29	32	33
7	0	8	39	72	88	91
8	0	14	94	230	315	342
9	0	21	222	700	1134	1305
10	0	39	572	2421	4691	5940
11	0	62	1463	8308	19838	28227
12	0	112	3934	30070	88467	144545
13	0	189	10584	108990	399679	760109
14	0	352	29211	403188	1839857	4112455
15	0	607	80808	1497057	8533474	22571026
16	0	1144	226430	5607200	39893579	125410006
17	0	2055	636320	21076570	187393549	702370207
18	0	3885	1800318	79595257	884152226	3959138462
19	0	7154	5107479	301492044	4185740194	22425417823
20	0	13602	14548360	1145558141	19876589828	127530807883
21	0	25472	41538874	4363503609	94633345517	727630240442
22	0	48670	118927919	16660196142	451615299593	4163114784625
23	0	92204	341187047	63741248200	2159769331316	23876534534361
24	0	176770	980838750	244339616370	10348546459872	137228556011456
25	0	337590	2824561080	938255682540	49672000435712	790200525479694
26	0	649341	8147547156	3608668279965	238804870806677	4557943660168122
27	0	1246840	23536592010	13900021843855	1149792978953236	26331300028827092
28	0	2404872	68087313892	53614339995060	5543621480301561	152331948562860704
29	0	4636389	197216119544	207062143625710	26762242828695895	882422888943285301
30	0	8964143	571924673368	800640167895069	129350646956169934	5117852381619320891
31	0	17334800	1660419530055	3099251723043168	625889663596316104	29715786751698562813

**Table A.32.** Total number of primitive inequivalent sequence structures with exactly  $q$  different characters under the action of  $D_n \times S_q$  where  $D_n$  is the dihedral group of cyclic shifts and reversals.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	1	1	0	0	0
4	0	2	2	1	0	0
5	0	3	5	2	1	0
6	0	5	13	11	3	1
7	0	8	31	33	16	3
8	0	14	80	136	85	27
9	0	21	201	478	434	171
10	0	39	533	1849	2270	1249
11	0	62	1401	6845	11530	8389
12	0	112	3822	26136	58397	56078
13	0	189	10395	98406	290689	360430
14	0	352	28859	373977	1436669	2272598
15	0	607	80201	1416249	7036417	14037552
16	0	1144	225286	5380770	34286379	85516427
17	0	2055	634265	20440250	166316979	514976658
18	0	3885	1796433	77794939	804556969	3074986236
19	0	7154	5100325	296384565	3884248150	18239677629
20	0	13602	14534758	1131009781	18731031687	107654218055
21	0	25472	41513402	4321964735	90269841908	632996894925
22	0	48670	118879249	16541268223	434955103451	3711499485032
23	0	92204	341094843	63400061153	2096028083116	21716765203045
24	0	176770	980661980	243358777620	10104206843502	126880009551584
25	0	337590	2824223490	935431121460	48733744753172	740528525043982
26	0	649341	8146897815	3600520732809	235196202526712	4319138789361445
27	0	1246840	23535345170	13876485251845	1135892957109381	25181507049873856
28	0	2404872	68084909020	53546252681168	5490007140306501	146788327082559143
29	0	4636389	197211483155	206864927506166	26555180685070185	855660646114589406
30	0	8964143	571915709225	800068243221701	128550006788274865	4988501734663150957
31	0	17334800	1660402195255	3097591303513113	622790411873272936	29089897088102246709

**Table A.33.** Total number of inequivalent sequences with  $\leq q$  characters under the action of  $H_n \times I_q$  where  $H_n$  is the group of step shifts.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	4	9	16	25	36
3	1	6	18	40	75	126
4	1	12	54	160	375	756
5	1	12	72	280	825	2016
6	1	40	405	2176	8125	23976
7	1	28	390	2800	13175	46956
8	1	96	1944	17920	103125	435456
9	1	104	3411	44224	327125	1683576
10	1	280	14985	263296	2445625	15128856
11	1	216	17802	419872	4884435	36284472
12	1	1248	139968	4280320	61640625	547204896
13	1	704	133104	5594000	101732425	1088416056
14	1	2800	798525	44751616	1017323125	13060989936
15	1	4344	1804518	134391040	3816215625	58782164616
16	1	8928	5454378	539054080	19104609375	352913845536
17	1	8232	8072532	1073758360	47683838325	1057916846196
18	1	44224	64599849	11453771776	635787765625	16926689693376
19	1	29204	64573626	15271054960	1059638680675	33853322280036
20	1	136032	437732424	137575813120	11924780390625	457078896068256
21	1	176752	872157294	366528038400	39736963221875	1828085963706576
22	1	419872	3138159429	1759220283904	238418603522125	13162170601924128
23	1	381492	4279259574	3198580043440	541860418146375	35896828419563076
24	1	2150400	35362084140	35193817661440	7451085205078125	592310845530464256
25	1	1678256	42364514403	56294998751872	14901161255281925	1421514402151564944
26	1	5594000	211822562025	375299991503616	124176343792385625	14215144021505563536
27	1	7461168	423646166250	1000800051424000	413921148003541875	56860576116456023856
28	1	22553408	1907275943628	6005166457282560	3104448333781328125	511747012892594033856

**Table A.34.** Total number of inequivalent sequences with exactly  $q$  different characters under the action of  $H_n \times I_q$  where  $H_n$  is the group of step shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	2	0	0	0	0
3	1	4	3	0	0	0
4	1	10	21	12	0	0
5	1	10	39	60	30	0
6	1	38	288	792	900	360
7	1	26	309	1404	2800	2520
8	1	94	1659	10716	32010	48060
9	1	102	3102	31200	139080	317520
10	1	278	14148	205032	1276200	4109040
11	1	214	17157	349956	2960940	12923136
12	1	1246	136227	3727932	41626230	238785300
13	1	702	130995	5065804	75086430	559279980
14	1	2798	790128	41574312	801522300	7612396920
15	1	4342	1791489	127199028	3162262170	37864711260
16	1	8926	5427597	517290132	16463793480	246263046840
17	1	8230	8047839	1041517620	42395689530	787758864480
18	1	44222	64467180	11195637720	579164463000	13282478342640
19	1	29202	64486017	15012935676	983928850100	27723264985920
20	1	136030	437324331	135825699612	11241277288950	387585098313300
21	1	176750	871627041	363040469732	37913042835300	1595144664456720
22	1	419870	3136899816	1746670165416	229653879498180	11757984528159432
23	1	381490	4278115101	3181465294092	525910306710000	32693559031867320
24	1	2150398	35355632943	35052382227276	7275469716108330	548131534355490300
25	1	1678254	42359479638	56125550763792	14620109889884040	1332951012336037248
26	1	5593998	211805780028	374452734819512	122301962004547800	13475711222256473520
27	1	7461166	423623782749	999105511526004	408921384133472700	54392052756394725120
28	1	22553406	1907208283407	5997537488828492	3074441574028817530	493210362241584732060

**Table A.35.** Total number of primitive inequivalent sequences with  $\leq q$  characters under the action of  $H_n \times I_q$  where  $H_n$  is the group of step shifts.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	2	6	12	20	30
3	0	4	15	36	70	120
4	0	8	45	144	350	720
5	0	10	69	276	820	2010
6	0	32	381	2124	8030	23820
7	0	26	387	2796	13170	46950
8	0	84	1890	17760	102750	434700
9	0	98	3393	44184	327050	1683450
10	0	266	14907	263004	2444780	15126810
11	0	214	17799	419868	4884430	36284466
12	0	1200	139518	4278000	61632150	547180200
13	0	702	133101	5593996	101732420	1088416050
14	0	2770	798129	44748804	1017309930	13060942950
15	0	4328	1804431	134390724	3816214730	58782162480
16	0	8832	5452434	539036160	19104506250	352913410080
17	0	8230	8072529	1073758356	47683838320	1057916846190
18	0	44086	64596051	11453725416	635787430450	16926687985950
19	0	29202	64573623	15271054956	1059638680670	33853322280030
20	0	135744	437717394	137575549680	11924777944650	457078880938680
21	0	176720	872156889	366528035564	39736963208630	1828085963659500
22	0	419654	3138141621	1759219864020	238418598637670	13162170565639626
23	0	381490	4279259571	3198580043436	541860418146370	35896828419563070
24	0	2149068	35361942282	35193813363360	7451085143334750	592310844982824660
25	0	1678244	42364514331	56294998751592	14901161255281100	1421514402151562928
26	0	5593294	211822428915	375299985909604	124176343690653180	14215144020417147450
27	0	7461064	423646162839	1000800051379776	413921148003214750	56860576116454340280
28	0	22550600	1907275145058	6005166412530800	3104448332764004650	511747012879533043200

**Table A.36.** Total number of primitive inequivalent sequences with exactly  $q$  different characters under the action of  $H_n \times I_q$  where  $H_n$  is the group of step shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	2	0	0	0	0
3	0	4	3	0	0	0
4	0	8	21	12	0	0
5	0	10	39	60	30	0
6	0	32	285	792	900	360
7	0	26	309	1404	2800	2520
8	0	84	1638	10704	32010	48060
9	0	98	3099	31200	139080	317520
10	0	266	14109	204972	1276170	4109040
11	0	214	17157	349956	2960940	12923136
12	0	1200	135918	3727128	41625330	238784940
13	0	702	130995	5065804	75086430	559279980
14	0	2770	789819	41572908	801519500	7612394400
15	0	4328	1791447	127198968	3162262140	37864711260
16	0	8832	5425938	517279416	16463761470	246262998780
17	0	8230	8047839	1041517620	42395689530	787758864480
18	0	44086	64463793	11195605728	579164323020	13282478024760
19	0	29202	64486017	15012935676	983928850100	27723264985920
20	0	135744	437310162	135825494568	11241276012750	387585094204260
21	0	176720	871626729	363040468328	37913042832500	1595144664454200
22	0	419654	3136882659	1746669815460	229653876537240	11757984515236296
23	0	381490	4278115101	3181465294092	525910306710000	32693559031867320
24	0	2149068	35355495078	35052378488640	7275469674450090	548131534116656940
25	0	1678244	42359479599	56125550763732	14620109889884010	1332951012336037248
26	0	5593294	211805649033	374452729753708	122301961929461370	13475711221697193540
27	0	7461064	423623779647	999105511494804	408921384133333620	54392052756394407600
28	0	22550600	1907207493258	5997537447254168	3074441573227295230	493210362233972335140

**Table A.37.** Total number of inequivalent sequence structures with  $\leq q$  characters under the action of  $H_n \times S_q$  where  $H_n$  is the group of step shifts.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	3	4	4	4	4
4	1	6	10	11	11	11
5	1	6	14	17	18	18
6	1	20	70	107	116	117
7	1	14	68	131	157	161
8	1	48	332	811	1110	1193
9	1	52	577	1893	3076	3530
10	1	140	2510	11107	21808	27569
11	1	108	2980	17599	42335	60333
12	1	624	23372	179371	528612	863409
13	1	352	22218	233449	859502	1636609
14	1	1400	133150	1866057	8547036	19122031
15	1	2172	300964	5603787	31968397	84580125
16	1	4464	909382	22469291	159716545	501915567
17	1	4116	1345634	44744047	398048506	1492170609
18	1	22112	10767202	477262537	5303680564	23751685183
19	1	14602	10762820	636308685	8835749347	47340359591
20	1	68016	72957100	5732457131	99411986634	637742820401
21	1	88376	145362932	15272176697	331215914028	2546703019245
22	1	209936	523029526	73301054891	1987084129352	18317620839483
23	1	190746	713213956	133274359129	4515860983385	49923584451715
24	1	1075200	5893709440	1466413263531	62095481040014	823391580631117
25	1	839128	7060765733	2345625787115	124179879021926	1975500747528184
26	1	2797000	35303782550	15637502446617	1034820535564128	19751086209723159
27	1	3730584	70607738788	41700005872963	3449378230680497	78993896078792369
28	1	11276704	317879412632	250215291427641	25870564475880830	710885078350879021
29	1	9587580	408514321970	428914261411367	55436071887736836	1827875967259171169

**Table A.38.** Total number of inequivalent sequence structures with exactly  $q$  different characters under the action of  $H_n \times S_q$  where  $H_n$  is the group of step shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	1	0	0	0
4	1	5	4	1	0	0
5	1	5	8	3	1	0
6	1	19	50	37	9	1
7	1	13	54	63	26	4
8	1	47	284	479	299	83
9	1	51	525	1316	1183	454
10	1	139	2370	8597	10701	5761
11	1	107	2872	14619	24736	17998
12	1	623	22748	155999	349241	334797
13	1	351	21866	211231	626053	777107
14	1	1399	131750	1732907	6680979	10574995
15	1	2171	298792	5302823	26364610	52611728
16	1	4463	904918	21559909	137247254	342199022
17	1	4115	1341518	43398413	353304459	1094122103
18	1	22111	10745090	466495335	4826418027	18448004619
19	1	14601	10748218	625545865	8199440662	38504610244
20	1	68015	72889084	5659500031	93679529503	538330833767
21	1	88375	145274556	15126813765	315943737331	2215487105217
22	1	209935	522819590	72778025365	1913783074461	16330536710131
23	1	190745	713023210	132561145173	4382586624256	45407723468330
24	1	1075199	5892634240	1460519554091	60629067776483	761296099591103
25	1	839127	7059926605	2338565021382	121834253234811	1851320868506258
26	1	2796999	35300985550	15602198664067	1019183033117511	18716265674159031
27	1	3730583	70604008204	41629398134175	3407678224807534	75544517848111872
28	1	11276703	317868135928	249897412015009	25620349184453189	685014513874998191
29	1	9587579	408504734390	428505747089397	55007157626325469	1772439895371434333

**Table A.39.** Total number of primitive inequivalent sequence structures with  $\leq q$  characters under the action of  $H_n \times S_q$  where  $H_n$  is the group of step shifts.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	0	2	3	3	3	3
4	0	4	8	9	9	9
5	0	5	13	16	17	17
6	0	16	65	102	111	112
7	0	13	67	130	156	160
8	0	42	322	800	1099	1182
9	0	49	573	1889	3072	3526
10	0	133	2495	11089	21789	27550
11	0	107	2979	17598	42334	60332
12	0	600	23294	179255	528487	863283
13	0	351	22217	233448	859501	1636608
14	0	1385	133081	1865925	8546878	19121869
15	0	2164	300947	5603767	31968376	84580104
16	0	4416	909050	22468480	159715435	501914374
17	0	4115	1345633	44744046	398048505	1492170608
18	0	22043	10766559	477260541	5303677376	23751681540
19	0	14601	10762819	636308684	8835749346	47340359590
20	0	67872	72954582	5732446015	99411964817	637742792823
21	0	88360	145362861	15272176563	331215913868	2546703019081
22	0	209827	523026545	73301037291	1987084087016	18317620779149
23	0	190745	713213955	133274359128	4515860983384	49923584451714
24	0	1074534	5893685746	1466413083360	62095480510303	823391579766526
25	0	839122	7060765719	2345625787098	124179879021908	1975500747528166
26	0	2796647	35303760331	15637502213167	1034820534704625	19751086208086549
27	0	3730532	70607738211	41700005871070	3449378230677421	78993896078788839
28	0	11275300	317879279474	250215289561575	25870564467333785	710885078331756981
29	0	9587579	408514321969	428914261411366	55436071887736835	1827875967259171168

**Table A.40.** Total number of primitive inequivalent sequence structures with exactly  $q$  different characters under the action of  $H_n \times S_q$  where  $H_n$  is the group of step shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	1	0	0	0
4	0	4	4	1	0	0
5	0	5	8	3	1	0
6	0	16	49	37	9	1
7	0	13	54	63	26	4
8	0	42	280	478	299	83
9	0	49	524	1316	1183	454
10	0	133	2362	8594	10700	5761
11	0	107	2872	14619	24736	17998
12	0	600	22694	155961	349232	334796
13	0	351	21866	211231	626053	777107
14	0	1385	131696	1732844	6680953	10574991
15	0	2164	298783	5302820	26364609	52611728
16	0	4416	904634	21559430	137246955	342198939
17	0	4115	1341518	43398413	353304459	1094122103
18	0	22043	10744516	466493982	4826416835	18448004164
19	0	14601	10748218	625545865	8199440662	38504610244
20	0	67872	72886710	5659491433	93679518802	538330828006
21	0	88360	145274501	15126813702	315943737305	2215487105213
22	0	209827	522816718	72778010746	1913783049725	16330536692133
23	0	190745	713023210	132561145173	4382586624256	45407723468330
24	0	1074534	5892611212	1460519397614	60629067426943	761296099256223
25	0	839122	7059926597	2338565021379	121834253234810	1851320868506258
26	0	2796647	35300963684	15602198452836	1019183032491458	18716265673381924
27	0	3730532	70604007679	41629398132859	3407678224806351	75544517848111418
28	0	11275300	317868004174	249897410282101	25620349177772210	685014513864423196
29	0	9587579	408504734390	428505747089397	55007157626325469	1772439895371434333

**Table A.41.** Total number of inequivalent sequences with  $\leq q$  characters under the action of  $E_n \times I_q$  where  $E_n$  is the group of step cyclic shifts.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	10	20	35	56
4	1	6	21	55	120	231
5	1	6	24	76	201	462
6	1	13	92	430	1505	4291
7	1	10	78	460	2015	6966
8	1	24	327	2605	14070	57561
9	1	22	443	5164	37085	188866
10	1	45	1632	26962	246753	1519035
11	1	30	1698	38572	445515	3302922
12	1	158	12769	367645	5205790	45921281
13	1	74	10464	431780	7832185	83747286
14	1	245	57840	3203430	72703645	933081411
15	1	368	122822	8993804	254689657	3920355712
16	1	693	348222	33860125	1196213445	22075451286
17	1	522	476052	63177820	2805046965	62230996506
18	1	2637	3597442	636462350	35322811755	940379310731
19	1	1610	3401970	803796700	55770979195	1781757016326
20	1	7386	22006959	6886280971	596439735024	22856965214727
21	1	8868	41597374	17456594380	1892294578755	87052415641136
22	1	19401	142677588	79965550558	10837223014665	598280600648031
23	1	16770	186077886	139069427020	23559159229935	1560731765058606
24	1	94484	1476697627	1466861706095	310484619147940	24680195365765751
25	1	67562	1694658003	2251803181492	596046508875701	56860576713326910
26	1	216275	8147282460	14434628481170	4776013513099405	546736312124316741
27	1	277534	15690973754	37066691779180	15330413466776835	2105947271634851386
28	1	815558	68149816689	214483458079665	110874578286500410	18276744394666643541
29	1	662370	84520682160	354963555781060	229389797793261945	45376420320719650686
30	1	4500267	857935531804	4803855154772166	3880512011730551301	921141380210973567989

**Table A.42.** Total number of inequivalent sequences with exactly  $q$  different characters under the action of  $E_n \times I_q$  where  $E_n$  is the group of step cyclic shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	1	0	0	0
4	1	4	6	3	0	0
5	1	4	9	12	6	0
6	1	11	56	136	150	60
7	1	8	51	204	400	360
8	1	22	258	1437	4080	6030
9	1	20	380	3520	15480	35280
10	1	43	1500	20700	127818	410976
11	1	28	1611	31956	269340	1174896
12	1	156	12298	317513	3493680	19948200
13	1	72	10245	390364	5777190	43022700
14	1	243	57108	2973536	57262450	543757860
15	1	366	121721	8504720	210945182	2524673904
16	1	691	346146	32471391	1030388115	15399118440
17	1	520	474489	61276740	2493913170	46338868800
18	1	2635	3589534	622088400	32176448060	737917466160
19	1	1608	3397143	790198476	51785999300	1459120076400
20	1	7384	21984804	6798297447	562228325904	19381180990752
21	1	8866	41570773	17290258088	1805427491920	75959665269840
22	1	19399	142619388	79394956608	10438821843750	534453892557660
23	1	16768	186027579	138325216092	22865672706000	1421459129778120
24	1	94482	1476414178	1460955482487	303165076648900	22839261043934250
25	1	67560	1694455320	2245024954848	584804438872656	53318040815648448
26	1	216273	8146633638	14402040648976	4703921841355410	518296587530532780
27	1	277532	15690141155	37003929549364	15145236914843140	2014520477395566000
28	1	815556	68147370018	214210863706253	109802842486113400	17614712813834735640
29	1	662368	84518695053	354625477026636	227615825214554550	44005404296893087260
30	1	4500265	857922031006	4800423439646548	3856501315267005846	897930348809268710592

**Table A.43.** Total number of primitive inequivalent sequences with  $\leq q$  characters under the action of  $E_n \times I_q$  where  $E_n$  is the group of step cyclic shifts.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	1	3	6	10	15
3	0	2	7	16	30	50
4	0	3	15	45	105	210
5	0	4	21	72	196	456
6	0	8	79	404	1460	4220
7	0	8	75	456	2010	6960
8	0	18	306	2550	13950	57330
9	0	18	433	5144	37050	188810
10	0	38	1605	26880	246542	1518558
11	0	28	1695	38568	445510	3302916
12	0	142	12662	367170	5204180	45916780
13	0	72	10461	431776	7832180	83747280
14	0	234	57759	3202964	72701620	933074430
15	0	360	122791	8993712	254689426	3920355200
16	0	669	347895	33857520	1196199375	22075393725
17	0	520	476049	63177816	2805046960	62230996500
18	0	2606	3596917	636456776	35322773200	940379117630
19	0	1608	3401967	803796696	55770979190	1781757016320
20	0	7338	22005312	6886253964	596439488166	22856963695482
21	0	8856	41597289	17456593904	1892294576710	87052415634120
22	0	19370	142675887	79965511980	10837222569140	598280597345094
23	0	16768	186077883	139069427016	23559159229930	1560731765058600
24	0	94308	1476684552	1466861335900	310484613928200	24680195319787140
25	0	67556	1694657979	2251803181416	596046508875500	56860576713326448
26	0	216200	8147271993	14434628049384	4776013505267210	546736312040569440
27	0	277512	15690973311	37066691774016	15330413466739750	2105947271634662520
28	0	815310	68149758834	214483454876190	110874578213796660	18276744393733561920
29	0	662368	84520682157	354963555781056	229389797793261940	45376420320719650680
30	0	4499852	857935407295	4803855145751072	3880512011475613632	921141380207051689484

**Table A.44.** Total number of primitive inequivalent sequences with exactly  $q$  different characters under the action of  $E_n \times I_q$  where  $E_n$  is the group of step cyclic shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	1	0	0	0
4	0	3	6	3	0	0
5	0	4	9	12	6	0
6	0	8	55	136	150	60
7	0	8	51	204	400	360
8	0	18	252	1434	4080	6030
9	0	18	379	3520	15480	35280
10	0	38	1491	20688	127812	410976
11	0	28	1611	31956	269340	1174896
12	0	142	12236	317374	3493530	19948140
13	0	72	10245	390364	5777190	43022700
14	0	234	57057	2973332	57262050	543757500
15	0	360	121711	8504708	210945176	2524673904
16	0	669	345888	32469954	1030384035	15399112410
17	0	520	474489	61276740	2493913170	46338868800
18	0	2606	3589099	622084744	32176432430	737917430820
19	0	1608	3397143	790198476	51785999300	1459120076400
20	0	7338	21983298	6798276744	562228198086	19381180579776
21	0	8856	41570721	17290257884	1805427491520	75959665269480
22	0	19370	142617777	79394924652	10438821574410	534453891382764
23	0	16768	186027579	138325216092	22865672706000	1421459129778120
24	0	94308	1476401628	1460955163540	303165073151140	22839261023980020
25	0	67556	1694455311	2245024954836	584804438872650	53318040815648448
26	0	216200	8146623393	14402040258612	4703921835578220	518296587487510080
27	0	277512	15690140775	37003929545844	15145236914827660	2014520477395530720
28	0	815310	68147312904	214210860732714	109802842428850950	17614712813290977780
29	0	662368	84518695053	354625477026636	227615825214554550	44005404296893087260
30	0	4499852	857921907739	4800423431121004	3856501315055932702	897930348806743625652

**Table A.45.** Total number of inequivalent sequence structures with  $\leq q$  characters under the action of  $E_n \times S_q$  where  $E_n$  is the group of step cyclic shifts.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	4	6	7	7	7
5	1	3	6	7	8	8
6	1	8	22	33	36	37
7	1	5	16	29	35	36
8	1	14	66	149	201	219
9	1	11	83	245	393	452
10	1	26	300	1230	2370	3000
11	1	15	296	1665	3971	5652
12	1	88	2243	15973	46094	74961
13	1	37	1778	18199	66675	126758
14	1	130	9780	134482	613439	1371016
15	1	184	20640	376433	2139350	5653854
16	1	357	58428	1415209	10016839	31440841
17	1	261	79554	2634597	23424253	87796362
18	1	1346	600798	26534612	294723254	1319722484
19	1	805	567544	33499359	465082931	2491714134
20	1	3760	3671978	287002711	4972761822	31893717065
21	1	4434	6935746	727425501	15773188813	121274777228
22	1	9758	23785912	3332040944	90323063945	832622962686
23	1	8385	31017008	5794658931	196342666487	2170594048582
24	1	47462	246156513	61120855045	2587522663248	34308978981089
25	1	33781	282456333	93825569009	4967200045397	79020052551081
26	1	108330	1357927400	601444738252	39800811900389	759657276910440
27	1	138767	2615206733	1544447425394	127754778614167	2925700012179146
28	1	408376	11358512518	8936822739237	923960848699762	25388849366234859
29	1	331185	14086865850	14790153116917	1911588773482017	63030206353101206
30	1	2251816	142990130376	200160698588045	32337673468535019	1279463184092998417
31	1	1155735	110694637376	206616781571371	4172597757333147	1981052450113962128

**Table A.46.** Total number of inequivalent sequence structures with exactly  $q$  different characters under the action of  $E_n \times S_q$  where  $E_n$  is the group of step cyclic shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	3	2	1	0	0
5	1	2	3	1	1	0
6	1	7	14	11	3	1
7	1	4	11	13	6	1
8	1	13	52	83	52	18
9	1	10	72	162	148	59
10	1	25	274	930	1140	630
11	1	14	281	1369	2306	1681
12	1	87	2155	13730	30121	28867
13	1	36	1741	16421	48476	60083
14	1	129	9650	124702	478957	757577
15	1	183	20456	355793	1762917	3514504
16	1	356	58071	1356781	8601630	21424002
17	1	260	79293	2555043	20789656	64372109
18	1	1345	599452	25933814	268188642	1024999230
19	1	804	566739	32931815	431583572	2026631203
20	1	3759	3668218	283330733	4685759111	26920955243
21	1	4433	6931312	720489755	15045763312	105501588415
22	1	9757	23776154	3308255032	86991023001	742299898741
23	1	8384	31008623	5763641923	190548007556	1974251382095
24	1	47461	246109051	60874698532	2526401808203	31721456317841
25	1	33780	282422552	93543112676	4873374476388	74052852505684
26	1	108329	1357819070	600086810852	39199367162137	719856465010051
27	1	138766	2615067966	1541832218661	126210331188773	2797945233564979
28	1	408375	11358104142	8925464226719	915024025960525	24464888517535097
29	1	331184	14086534665	14776066251067	1896798620365100	61118617579619189
30	1	2251815	142987878560	200017708457669	32137512769946974	1247125510624463398
31	1	1155734	110693481641	206506086933995	41519360791761776	1939326472540628981

**Table A.47.** Total number of primitive inequivalent sequence structures with  $\leq q$  characters under the action of  $E_n \times S_q$  where  $E_n$  is the group of step cyclic shifts.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	0	1	2	2	2	2
4	0	2	4	5	5	5
5	0	2	5	6	7	7
6	0	5	18	29	32	33
7	0	4	15	28	34	35
8	0	10	60	142	194	212
9	0	9	80	242	390	449
10	0	22	293	1222	2361	2991
11	0	14	295	1664	3970	5651
12	0	78	2217	15935	46053	74919
13	0	36	1777	18198	66674	126757
14	0	124	9763	134452	613403	1370979
15	0	180	20632	376424	2139340	5653844
16	0	343	58362	1415060	10016638	31440622
17	0	260	79553	2634596	23424252	87796361
18	0	1329	600696	26534337	294722828	1319721998
19	0	804	567543	33499358	465082930	2491714133
20	0	3732	3671674	287001476	4972759447	31893714060
21	0	4428	6935728	727425470	15773188776	121274777190
22	0	9742	23785615	3332039278	90323059973	832622957033
23	0	8384	31017007	5794658930	196342666486	2170594048581
24	0	47364	246154210	61120838930	2587522616960	34308978905916
25	0	33778	282456327	93825569002	4967200045389	79020052551073
26	0	108292	1357925621	601444720052	39800811833713	759657276783681
27	0	138756	2615206650	1544447425149	127754778613774	2925700012178694
28	0	408244	11358502734	8936822604750	923960848086318	25388849364863838
29	0	331184	14086865849	14790153116916	1911588773482016	63030206353101205
30	0	2251604	142990109424	200160698210360	32337673466393275	1279463184087341538
31	0	1155734	110694637375	206616781571370	41725977573333146	1981052450113962127

**Table A.48.** Total number of primitive inequivalent sequence structures with exactly  $q$  different characters under the action of  $E_n \times S_q$  where  $E_n$  is the group of step cyclic shifts.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	1	1	0	0	0
4	0	2	2	1	0	0
5	0	2	3	1	1	0
6	0	5	13	11	3	1
7	0	4	11	13	6	1
8	0	10	50	82	52	18
9	0	9	71	162	148	59
10	0	22	271	929	1139	630
11	0	14	281	1369	2306	1681
12	0	78	2139	13718	30118	28866
13	0	36	1741	16421	48476	60083
14	0	124	9639	124689	478951	757576
15	0	180	20452	355792	1762916	3514504
16	0	343	58019	1356698	8601578	21423984
17	0	260	79293	2555043	20789656	64372109
18	0	1329	599367	25933641	268188491	1024999170
19	0	804	566739	32931815	431583572	2026631203
20	0	3732	3667942	283329802	4685757971	26920954613
21	0	4428	6931300	720489742	15045763306	105501588414
22	0	9742	23775873	3308253663	86991020695	742299897060
23	0	8384	31008623	5763641923	190548007556	1974251382095
24	0	47364	246106846	60874684720	2526401778030	31721456288956
25	0	33778	282422549	93543112675	4873374476387	74052852505684
26	0	108292	1357817329	600086794431	39199367113661	719856464949968
27	0	138756	2615067894	1541832218499	126210331188625	2797945233564920
28	0	408244	11358094490	8925464102016	915024025481568	24464888516777520
29	0	331184	14086534665	14776066251067	1896798620365100	61118617579619189
30	0	2251604	142987857820	200017708100936	32137512768182915	1247125510620948263
31	0	1155734	110693481641	206506086933995	41519360791761776	1939326472540628981

**Table A.49.** Total number of finite palindromes with  $\leq q$  characters.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	2	3	4	5	6
3	1	4	9	16	25	36
4	1	4	9	16	25	36
5	1	8	27	64	125	216
6	1	8	27	64	125	216
7	1	16	81	256	625	1296
8	1	16	81	256	625	1296
9	1	32	243	1024	3125	7776
10	1	32	243	1024	3125	7776
11	1	64	729	4096	15625	46656
12	1	64	729	4096	15625	46656
13	1	128	2187	16384	78125	279936
14	1	128	2187	16384	78125	279936
15	1	256	6561	65536	390625	1679616
16	1	256	6561	65536	390625	1679616
17	1	512	19683	262144	1953125	10077696
18	1	512	19683	262144	1953125	10077696
19	1	1024	59049	1048576	9765625	60466176
20	1	1024	59049	1048576	9765625	60466176
21	1	2048	177147	4194304	48828125	362797056
22	1	2048	177147	4194304	48828125	362797056
23	1	4096	531441	16777216	244140625	2176782336
24	1	4096	531441	16777216	244140625	2176782336
25	1	8192	1594323	67108864	1220703125	13060694016
26	1	8192	1594323	67108864	1220703125	13060694016
27	1	16384	4782969	268435456	6103515625	78364164096
28	1	16384	4782969	268435456	6103515625	78364164096
29	1	32768	14348907	1073741824	30517578125	470184984576
30	1	32768	14348907	1073741824	30517578125	470184984576
31	1	65536	43046721	4294967296	152587890625	2821109907456

**Table A.50.** Total number of finite palindromes with exactly  $q$  different characters.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	0	0	0	0	0
3	1	2	0	0	0	0
4	1	2	0	0	0	0
5	1	6	6	0	0	0
6	1	6	6	0	0	0
7	1	14	36	24	0	0
8	1	14	36	24	0	0
9	1	30	150	240	120	0
10	1	30	150	240	120	0
11	1	62	540	1560	1800	720
12	1	62	540	1560	1800	720
13	1	126	1806	8400	16800	15120
14	1	126	1806	8400	16800	15120
15	1	254	5796	40824	126000	191520
16	1	254	5796	40824	126000	191520
17	1	510	18150	186480	834120	1905120
18	1	510	18150	186480	834120	1905120
19	1	1022	55980	818520	5103000	16435440
20	1	1022	55980	818520	5103000	16435440
21	1	2046	171006	3498000	29607600	129230640
22	1	2046	171006	3498000	29607600	129230640
23	1	4094	519156	14676024	165528000	953029440
24	1	4094	519156	14676024	165528000	953029440
25	1	8190	1569750	60780720	901020120	6711344640
26	1	8190	1569750	60780720	901020120	6711344640
27	1	16382	4733820	249401880	4809004200	45674188560
28	1	16382	4733820	249401880	4809004200	45674188560
29	1	32766	14250606	1016542800	25292030400	302899156560
30	1	32766	14250606	1016542800	25292030400	302899156560
31	1	65534	42850116	4123173624	131542866000	1969147121760

**Table A.51.** Total number of primitive finite palindromes with  $\leq q$  characters.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	0	0	0	0	0
3	0	2	6	12	20	30
4	0	2	6	12	20	30
5	0	6	24	60	120	210
6	0	4	18	48	100	180
7	0	14	78	252	620	1290
8	0	12	72	240	600	1260
9	0	28	234	1008	3100	7740
10	0	24	216	960	3000	7560
11	0	62	726	4092	15620	46650
12	0	54	696	4020	15480	46410
13	0	126	2184	16380	78120	279930
14	0	112	2106	16128	77500	278640
15	0	246	6528	65460	390480	1679370
16	0	240	6480	65280	390000	1678320
17	0	510	19680	262140	1953120	10077690
18	0	476	19422	261072	1949900	10069740
19	0	1022	59046	1048572	9765620	60466170
20	0	990	58800	1047540	9762480	60458370
21	0	2030	177060	4194036	48827480	362795730
22	0	1984	176418	4190208	48812500	362750400
23	0	4094	531438	16777212	244140620	2176782330
24	0	4020	530640	16772880	244124400	2176734420
25	0	8184	1594296	67108800	1220703000	13060693800
26	0	8064	1592136	67092480	1220625000	13060414080
27	0	16352	4782726	268434432	6103512500	78364156320
28	0	16254	4780776	268419060	6103437480	78363884130
29	0	32766	14348904	1073741820	30517578120	470184984570
30	0	32484	14342112	1073675280	30517184400	470183297220
31	0	65534	43046718	4294967292	152587890620	2821109907450

**Table A.52.** Total number of primitive finite palindromes with exactly  $q$  different characters.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	0	0	0	0	0
3	0	2	0	0	0	0
4	0	2	0	0	0	0
5	0	6	6	0	0	0
6	0	4	6	0	0	0
7	0	14	36	24	0	0
8	0	12	36	24	0	0
9	0	28	150	240	120	0
10	0	24	144	240	120	0
11	0	62	540	1560	1800	720
12	0	54	534	1560	1800	720
13	0	126	1806	8400	16800	15120
14	0	112	1770	8376	16800	15120
15	0	246	5790	40824	126000	191520
16	0	240	5760	40800	126000	191520
17	0	510	18150	186480	834120	1905120
18	0	476	17994	186240	834000	1905120
19	0	1022	55980	818520	5103000	16435440
20	0	990	55830	818280	5102880	16435440
21	0	2030	170970	3497976	29607600	129230640
22	0	1984	170466	3496440	29605800	129229920
23	0	4094	519156	14676024	165528000	953029440
24	0	4020	518580	14674440	165526200	953028720
25	0	8184	1569744	60780720	901020120	6711344640
26	0	8064	1567944	60772320	901003320	6711329520
27	0	16352	4733670	249401640	4809004080	45674188560
28	0	16254	4732014	249393480	4808987400	45674173440
29	0	32766	14250606	1016542800	25292030400	302899156560
30	0	32484	14244660	1016501736	25291904280	302898965040
31	0	65534	42850116	4123173624	131542866000	1969147121760

**Table A.53.** Total number of finite palindromic structures with  $\leq q$  characters.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	2	2	2	2	2
4	1	2	2	2	2	2
5	1	4	5	5	5	5
6	1	4	5	5	5	5
7	1	8	14	15	15	15
8	1	8	14	15	15	15
9	1	16	41	51	52	52
10	1	16	41	51	52	52
11	1	32	122	187	202	203
12	1	32	122	187	202	203
13	1	64	365	715	855	876
14	1	64	365	715	855	876
15	1	128	1094	2795	3845	4111
16	1	128	1094	2795	3845	4111
17	1	256	3281	11051	18002	20648
18	1	256	3281	11051	18002	20648
19	1	512	9842	43947	86472	109299
20	1	512	9842	43947	86472	109299
21	1	1024	29525	175275	422005	601492
22	1	1024	29525	175275	422005	601492
23	1	2048	88574	700075	2079475	3403127
24	1	2048	88574	700075	2079475	3403127
25	1	4096	265721	2798251	10306752	19628064
26	1	4096	265721	2798251	10306752	19628064
27	1	8192	797162	11188907	51263942	114700315
28	1	8192	797162	11188907	51263942	114700315
29	1	16384	2391485	44747435	255514355	676207628
30	1	16384	2391485	44747435	255514355	676207628
31	1	32768	7174454	178973355	1275163905	4010090463

**Table A.54.** Total number of finite palindromic structures with exactly  $q$  different characters.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	0	0	0	0	0
3	1	1	0	0	0	0
4	1	1	0	0	0	0
5	1	3	1	0	0	0
6	1	3	1	0	0	0
7	1	7	6	1	0	0
8	1	7	6	1	0	0
9	1	15	25	10	1	0
10	1	15	25	10	1	0
11	1	31	90	65	15	1
12	1	31	90	65	15	1
13	1	63	301	350	140	21
14	1	63	301	350	140	21
15	1	127	966	1701	1050	266
16	1	127	966	1701	1050	266
17	1	255	3025	7770	6951	2646
18	1	255	3025	7770	6951	2646
19	1	511	9330	34105	42525	22827
20	1	511	9330	34105	42525	22827
21	1	1023	28501	145750	246730	179487
22	1	1023	28501	145750	246730	179487
23	1	2047	86526	611501	1379400	1323652
24	1	2047	86526	611501	1379400	1323652
25	1	4095	261625	2532530	7508501	9321312
26	1	4095	261625	2532530	7508501	9321312
27	1	8191	788970	10391745	40075035	63436373
28	1	8191	788970	10391745	40075035	63436373
29	1	16383	2375101	42355950	210766920	420693273
30	1	16383	2375101	42355950	210766920	420693273
31	1	32767	7141686	171798901	1096190550	2734926558

**Table A.55.** Total number of primitive finite palindromic structures with  $\leq q$  characters.

n	q					
	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	0	0	0	0	0
3	0	1	1	1	1	1
4	0	1	1	1	1	1
5	0	3	4	4	4	4
6	0	2	3	3	3	3
7	0	7	13	14	14	14
8	0	6	12	13	13	13
9	0	14	39	49	50	50
10	0	12	36	46	47	47
11	0	31	121	186	201	202
12	0	27	116	181	196	197
13	0	63	364	714	854	875
14	0	56	351	700	840	861
15	0	123	1088	2789	3839	4105
16	0	120	1080	2780	3830	4096
17	0	255	3280	11050	18001	20647
18	0	238	3237	10997	17947	20593
19	0	511	9841	43946	86471	109298
20	0	495	9800	43895	86419	109246
21	0	1015	29510	175259	421989	601476
22	0	992	29403	175088	421803	601289
23	0	2047	88573	700074	2079474	3403126
24	0	2010	88440	699875	2079260	3402911
25	0	4092	265716	2798246	10306747	19628059
26	0	4032	265356	2797536	10305897	19627188
27	0	8176	797121	11188856	51263890	114700263
28	0	8127	796796	11188191	51263086	114699438
29	0	16383	2391484	44747434	255514354	676207627
30	0	16242	2390352	44744591	255510460	676203467
31	0	32767	7174453	178973354	1275163904	4010090462

**Table A.56.** Total number of primitive finite palindromic structures with exactly  $q$  different characters.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	0	0	0	0	0
3	0	1	0	0	0	0
4	0	1	0	0	0	0
5	0	3	1	0	0	0
6	0	2	1	0	0	0
7	0	7	6	1	0	0
8	0	6	6	1	0	0
9	0	14	25	10	1	0
10	0	12	24	10	1	0
11	0	31	90	65	15	1
12	0	27	89	65	15	1
13	0	63	301	350	140	21
14	0	56	295	349	140	21
15	0	123	965	1701	1050	266
16	0	120	960	1700	1050	266
17	0	255	3025	7770	6951	2646
18	0	238	2999	7760	6950	2646
19	0	511	9330	34105	42525	22827
20	0	495	9305	34095	42524	22827
21	0	1015	28495	145749	246730	179487
22	0	992	28411	145685	246715	179486
23	0	2047	86526	611501	1379400	1323652
24	0	2010	86430	611435	1379385	1323651
25	0	4092	261624	2532530	7508501	9321312
26	0	4032	261324	2532180	7508361	9321291
27	0	8176	788945	10391735	40075034	63436373
28	0	8127	788669	10391395	40074895	63436352
29	0	16383	2375101	42355950	210766920	420693273
30	0	16242	2374110	42354239	210765869	420693007
31	0	32767	7141686	171798901	1096190550	2734926558

**Table A.57.** Total number of periodic palindromes with  $\leq q$  characters.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	9	16	25	36
4	1	6	18	40	75	126
5	1	8	27	64	125	216
6	1	12	54	160	375	756
7	1	16	81	256	625	1296
8	1	24	162	640	1875	4536
9	1	32	243	1024	3125	7776
10	1	48	486	2560	9375	27216
11	1	64	729	4096	15625	46656
12	1	96	1458	10240	46875	163296
13	1	128	2187	16384	78125	279936
14	1	192	4374	40960	234375	979776
15	1	256	6561	65536	390625	1679616
16	1	384	13122	163840	1171875	5878656
17	1	512	19683	262144	1953125	10077696
18	1	768	39366	655360	5859375	35271936
19	1	1024	59049	1048576	9765625	60466176
20	1	1536	118098	2621440	29296875	211631616
21	1	2048	177147	4194304	48828125	362797056
22	1	3072	354294	10485760	146484375	1269789696
23	1	4096	531441	16777216	244140625	2176782336
24	1	6144	1062882	41943040	732421875	7618738176
25	1	8192	1594323	67108864	1220703125	13060694016
26	1	12288	3188646	167772160	3662109375	45712429056
27	1	16384	4782969	268435456	6103515625	78364164096
28	1	24576	9565938	671088640	18310546875	274274574336
29	1	32768	14348907	1073741824	30517578125	470184984576
30	1	49152	28697814	2684354560	91552734375	1645647446016
31	1	65536	43046721	4294967296	152587890625	2821109907456

**Table A.58.** Total number of periodic palindromes with exactly  $q$  different characters.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	0	0	0	0
4	1	4	3	0	0	0
5	1	6	6	0	0	0
6	1	10	21	12	0	0
7	1	14	36	24	0	0
8	1	22	93	132	60	0
9	1	30	150	240	120	0
10	1	46	345	900	960	360
11	1	62	540	1560	1800	720
12	1	94	1173	4980	9300	7920
13	1	126	1806	8400	16800	15120
14	1	190	3801	24612	71400	103320
15	1	254	5796	40824	126000	191520
16	1	382	11973	113652	480060	1048320
17	1	510	18150	186480	834120	1905120
18	1	766	37065	502500	2968560	9170280
19	1	1022	55980	818520	5103000	16435440
20	1	1534	113493	2158260	17355300	72833040
21	1	2046	171006	3498000	29607600	129230640
22	1	3070	345081	9087012	97567800	541130040
23	1	4094	519156	14676024	165528000	953029440
24	1	6142	1044453	37728372	533274060	3832187040
25	1	8190	1569750	60780720	901020120	6711344640
26	1	12286	3151785	155091300	2855012160	26192766600
27	1	16382	4733820	249401880	4809004200	45674188560
28	1	24574	9492213	632972340	15050517300	174286672560
29	1	32766	14250606	1016542800	25292030400	302899156560
30	1	49150	28550361	2569858212	78417448200	1136023139160
31	1	65534	42850116	4123173624	131542866000	1969147121760

**Table A.59.** Total number of primitive periodic palindromes with  $\leq q$  characters.

n	q					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	0	1	3	6	10	15
3	0	2	6	12	20	30
4	0	3	12	30	60	105
5	0	6	24	60	120	210
6	0	7	42	138	340	705
7	0	14	78	252	620	1290
8	0	18	144	600	1800	4410
9	0	28	234	1008	3100	7740
10	0	39	456	2490	9240	26985
11	0	62	726	4092	15620	46650
12	0	81	1392	10050	46440	162435
13	0	126	2184	16380	78120	279930
14	0	175	4290	40698	233740	978465
15	0	246	6528	65460	390480	1679370
16	0	360	12960	163200	1170000	5874120
17	0	510	19680	262140	1953120	10077690
18	0	728	39078	654192	5855900	35263440
19	0	1022	59046	1048572	9765620	60466170
20	0	1485	117600	2618850	29287440	211604295
21	0	2030	177060	4194036	48827480	362795730
22	0	3007	353562	10481658	146468740	1269743025
23	0	4094	531438	16777212	244140620	2176782330
24	0	6030	1061280	41932200	732373200	7618570470
25	0	8184	1594296	67108800	1220703000	13060693800
26	0	12159	3186456	167755770	3662031240	45712149105
27	0	16352	4782726	268434432	6103512500	78364156320
28	0	24381	9561552	671047650	18310312440	274273594455
29	0	32766	14348904	1073741820	30517578120	470184984570
30	0	48849	28690752	2684286390	91552334160	1645645738695
31	0	65534	43046718	4294967292	152587890620	2821109907450

**Table A.60.** Total number of primitive periodic palindromes with exactly  $q$  different characters.

$n$	$q$					
	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	2	0	0	0	0
4	0	3	3	0	0	0
5	0	6	6	0	0	0
6	0	7	21	12	0	0
7	0	14	36	24	0	0
8	0	18	90	132	60	0
9	0	28	150	240	120	0
10	0	39	339	900	960	360
11	0	62	540	1560	1800	720
12	0	81	1149	4968	9300	7920
13	0	126	1806	8400	16800	15120
14	0	175	3765	24588	71400	103320
15	0	246	5790	40824	126000	191520
16	0	360	11880	113520	480000	1048320
17	0	510	18150	186480	834120	1905120
18	0	728	36894	502248	2968440	9170280
19	0	1022	55980	818520	5103000	16435440
20	0	1485	113145	2157360	17354340	72832680
21	0	2030	170970	3497976	29607600	129230640
22	0	3007	344541	9085452	97566000	541129320
23	0	4094	519156	14676024	165528000	953029440
24	0	6030	1043190	37723260	533264700	3832179120
25	0	8184	1569744	60780720	901020120	6711344640
26	0	12159	3149979	155082900	2854995360	26192751480
27	0	16352	4733670	249401640	4809004080	45674188560
28	0	24381	9488409	632947728	15050445900	174286569240
29	0	32766	14250606	1016542800	25292030400	302899156560
30	0	48849	28544205	2569816476	78417321240	1136022947280
31	0	65534	42850116	4123173624	131542866000	1969147121760

**Table A.61.** Total number of periodic palindromic structures with  $\leq q$  characters.

These counts were obtained by performing computer searches, and empty cells indicate that the corresponding computer search was not initiated.

n	q					
	1	2	3	4	5	6
1	1					
2	1	2				
3	1	2	2			
4	1	4	5	5		
5	1	4	5	5	5	
6	1	7	12	13	13	13
7	1	8	14	15	15	15
8	1	14	33	40	41	41
9	1	16	41	51	52	52
10	1	26	90	133	143	144
11	1	32	122	187	202	203
12	1	51	259	479	564	577
13	1	64	365	715		
14	1	100				
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						
31						

**Table A.62.** Total number of periodic palindromic structures with exactly  $q$  different characters. These counts were obtained by performing computer searches, and empty cells indicate that the corresponding computer search was not initiated.

$n$	$q$					
	1	2	3	4	5	6
1	1					
2	1	1				
3	1	1	0			
4	1	3	1	0		
5	1	3	1	0	0	
6	1	6	5	1	0	0
7	1	7	6	1	0	0
8	1	13	19	7	1	0
9	1	15	25	10	1	0
10	1	25	64	43	10	1
11	1	31	90	65	15	1
12	1	50	208	220	85	13
13	1	63	301	350		
14	1	99				
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						
31						

**Table A.63.** Total number of primitive periodic palindromic structures with  $\leq q$  characters. These counts were obtained by performing computer searches, and empty cells indicate that the corresponding computer search was not initiated.

n	q					
	1	2	3	4	5	6
1	1					
2	0	1				
3	0	1	1			
4	0	2	3	3		
5	0	3	4	4	4	
6	0	4	9	10	10	10
7	0	7	13	14	14	14
8	0	10	28	35	36	36
9	0	14	39	49	50	50
10	0	21	84	127	137	138
11	0	31	121	186	201	202
12	0	42	244	463	548	561
13	0	63	364	714		
14	0	91				
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						
31						

**Table A.64.** Total number of primitive periodic palindromic structures with exactly  $q$  different characters. These counts were obtained by performing computer searches, and empty cells indicate that the corresponding computer search was not initiated.

$n$	$q$					
	1	2	3	4	5	6
1	1					
2	0	1				
3	0	1	0			
4	0	2	1	0		
5	0	3	1	0	0	
6	0	4	5	1	0	0
7	0	7	6	1	0	0
8	0	10	18	7	1	0
9	0	14	25	10	1	0
10	0	21	63	43	10	1
11	0	31	90	65	15	1
12	0	42	202	219	85	13
13	0	63	301	350		
14	0	91				
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						
31						

