CANONICAL POLYGONS

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Abstract

Canonical Polygons are plane polygons defined on a square lattice with limitations on length of sides. The smaller sets are enumerated, and metrical and non-metrical properties are defined and calculated.

1. INTRODUCTION

The idea which gave birth to the canonical polygons (hereafter designated as CPs) arose informally to the author, while drawing figures on graph paper. Special attention was given to a certain class of polygons whose sides followed the lattice lines or diagonals. Adopting the restriction that each side should not include more than one square division, whether orthogonal or diagonal, there arose the concept of canonical polygon – canonical because it is constructed according to simple and well-defined rules, which limit it as to extension, shape and number.
An alternative way of expressing the concept of CP is drawing a closed polygonal chain passing through the square lattice intersection points, proceeding from each point to one of the 8 that are adjacent to it in the orthogonal or diagonal direction, necessarily changing direction after each segment.
In January 1977, at the very beginning of the activity in experimental combinatorial geometry that created them, it was discovered that there exist only 8 convex CPs – a fact which is intuitively obvious from their graphical representations, but which can be formally proved, as shown below.

2. FORMAL DEFINITION

A **canonical *n*-gon** is defined as a plane, closed chain of n consecutive segments between points of integer coordinates $(x_1, y_1), \ldots, (x_n, y_n)$, such that the following provisos hold true:

(a) Each segment joins a point (x,y) to a point $(x\pm1,y\pm1)$;	**Conway neighbourhood** [analogous to the 8 neighbours of each cell in J.H. Conway's Game of Life]
(b) There exists a segment from (x,y) to $(x+\xi,y+\eta)$ $\leftrightarrow$ there exists no segment from $(x+\xi,y+\eta)$ to (x,y) or to $(x+2\xi,y+2\eta)$ for $\xi,\eta \in \{-1,0,1\}$;	**non-consecutive collinearity**
(c) There exists a segment from (x,y) to $(x+\xi',y+\eta')$ $\leftrightarrow$ there exists no segment from $(x,y+\eta')$ to $(x+\xi',y)$ for $\xi',\eta' \in \{-1,1\}$;	**non-crossing of sides**
(d) No point is the endpoint of more than two segments;	**singularity of vertices**
(e) Chains that differ only by translation, reflection or rotation in the plane are not considered distinct.	**topological freedom in the plane**

Thus, a canonical polygon is one whose sides are straight-line segments proportional to 1 (in two mutually perpendicular directions) and $\sqrt{2}$ (in two other directions, also mutually perpendicular, at ½ right angle to the previous ones), and whose interior angles are of the form $k/2$ right angles, where k = 1, 2, 3, 5, 6 or 7 (k = 4 is excluded because of proviso (b)). Cases of crossed sides or multiple vertices are excluded.

3. INFORMAL DEFINITION

A CP is a polygon that may be drawn on a plane square lattice in such a way that each one of its sides is a side or a diagonal of one of the lattice's squares; this excludes, as possible sides of the CP, segments that include more than one side or diagonal of those squares. Crossed sides and multiple vertices are excluded.

4. TERMINOLOGY

The short and long sides of CPs (whose lengths are proportional to 1 and to $\sqrt{2}$) are called **orthogonal** [**o**] and **diagonal** [**d**] sides respectively.
Two CPs that possess the same interior angles in the same order, with **o** and **d** sides interchanged, are said to be **dual** to one another. If such a **duality transformation** generates the same CP, it is said to be **auto-dual**.

5. PROPERTIES

Below are defined some properties that depend only upon the shape of the CP, and some others which involve measurements. The symbols for these properties refer to the *Canonical Polygon Catalogue* table, where *n* denotes the number of the CP's sides and # is its order among those of *n* sides.

5.1. Properties dependent on shape only

5.1.1. Interior angle formula [*Fa*]: An ordered sequence of the CP's interior angles, expressed as multiples of $\pi/4$, starting with the largest and in that direction which produces the largest numerical expression (concatenation of the values of *k* for consecutive angles, as defined above). It is followed by the letter(s) **o** (and/)or **d**, according to whether the segment following the largest interior angle, in the sense defined above, is orthogonal (and/)or diagonal.
This formula is useful for the textual description of a CP (without a graphical representation) and in constructing CPs for a given *n* without duplications.

5.1.2. Duality [*Du*]: **+**, **-** or **A** according to whether the CP is dual to the following one (on the list ordered by interior angle formulas), to the preceding one or auto-dual.

5.1.3. Number of concavities [*K*]: A concavity is a polygon bounded by the CP and the minimal-area convex polygon which circumscribes it (its **convex hull**).

5.1.4. Directional formula [*Fd*]: A sequence of four integers representing in order:

- the number of the CP's sides in the most frequent orthogonal direction;
- the number of the CP's sides in the least frequent orthogonal direction;
- the number of the CP's sides in the most frequent diagonal direction;
- the number of the CP's sides in the least frequent diagonal direction.

5.1.5. Symmetry [*Sm*]: **Ay**, if the CP has *y* axes of symmetry (axial); **C**, if it has only a centre of symmetry (central).

5.2. Properties dependent also on measure

5.2.1. Double area [*Da*]: Expressed as a multiple of the area of the fundamental lattice square, doubled for convenience.

5.2.2. Perimeter [P]: Expressed as a multiple of the side of the fundamental lattice square.
5.2.3. Diameter [D]: Maximal distance between CP vertices.
5.2.4. Area/diameter relation [A/D]: Quotient of ½ *Da* and *D* in the units given above.
5.2.5. Perimeter/diameter relation [P/D]: Quotient of *P* and *D* in the units given above.
5.2.6. Convexity [C]: Ratio of the CP's area to that of its convex hull.
5.2.7. Square fraction [Q]: Fraction of the CP's area constituted of lattice squares.
A larger value of *f* denotes a CP that is more “compact” and less “stretched”.

6. CONSTRUCTION

Two approaches have been examined for construction of all CPs with a given number of sides *n*:

- **Construction by Segments**: Traces all closed polygonal chains – under the canonical restrictions – with *n* segments(*).
- **Recursive Construction**: Constructs CPs of *n* sides starting with those of *n*-1 sides, adding to or subtracting from them conveniently located canonical triangles.

A way of constructing CPs

The following text is due to QUANDT [1] and appears here in translated and slightly edited form.

Each CP is a sequence of *n* objects from the set $\{1, -1, i, -i, 1+i, 1-i, -1+i, -1-i\}$, subject to some restrictions, such as:

- the sum of the sequence is zero (it is a closed circuit)
- the sum of any “fragment” of the sequence is non-zero (the circuit does not close in less than *n* steps)
- two consecutive elements neither are equal nor differ only in sign
- the grouping $\ldots, 1+i, -i, -1+i, \ldots$ is not allowed […]

(“fragment” is a grouping of consecutive terms)
Clearly two sequences are equivalent if:

a) they differ only in order of reading
b) they differ only in choice of 1st term (are read cyclically)

Furthermore two sequences are equivalent if one can be obtained from the other through one or more among the following operations:

- multiplication by 1, by -1, by *i* or by -*i*
- conjugation

Ex.:

$$(1+i, 1, -1+i, -1-i, i)$$
$$\xrightarrow{\times i} (-1+i, i, -1-i, 1-i, 1) \xrightarrow{\text{conjugate}} (-1-i, -i, -1+i, 1+i, 1) \xrightarrow{\times -1} (1+i, i, 1-i, -1-i, -1)$$

A sequence is equivalent to itself; if two sequences are equivalent to a third, they are so between themselves; if sequence A is equivalent to B, B is equivalent to A.
Thus a CP is the equivalence class of a sequence, *i. e.*, the set of equivalent sequences. To draw a CP it suffices to take a representative of the class.
Two CPs are dual if a representative sequence of the one can be converted into a representative sequence of the other through the following substitutions:

instead of	→	take
1		1+*i*
-1		-1-*i*
i		-1+*i*

-i		1-i
1+i		i
-1-i		-i
-1+i		-1
1-i		1
take	←	instead of

[…]
Obviously auto-duality is only a reflection and/or rotation; the dual of the dual is the same case.
[… A] good programmer will be able to set up an adequate sequence-generating program (I believe Pascal or C++ will do). It is "enough" to use advisedly the most sophisticated resources to eliminate equivalencies. An obvious but useful remark: the perimeter of a sequence $(a_1, a_2, \ldots a_n)$ is $\sum_{j=1}^{n}|a_j|$ and sequences of unequal perimeters are not equivalent.
Also obvious: one may always take $a_1 = 1$ and $a_2 = 1 + i$ [footnote: or $a_1 = 1$ and $a_2 = i$].
[…]
See how easy it is to determine the diameter: given a sequence $(a_1, a_2, \ldots a_n)$, construct the associated sequence $(b_1, b_2, \ldots b_n)$ by making $b_1 = 0, b_j = b_{j-1} + a_{j-1}$ $(2 \leq j \leq n)$.
Calculate the $n(n-1)/2$ differences $b_j - b_k$, $1 \leq j \leq n-1, j+1 \leq k \leq n$. The largest among the numbers $|b_j - b_k|$ is the diameter.

A conjecture that has been proven false is the following:
It is possible to construct all CPs of n sides starting with those of n-1 sides by the recursive process, for every n > 4.
A counterexample of this intuitively true-seeming conjecture was given by STOLFI [2], whose following text has been translated and very slightly redacted.

If the restriction against 180-degree angles [at vertices] is valid, I think I have a counterexample for the [conjecture] on recursive construction. The vertices are

(1,0) (2,1) (3,1) (2,2) (2,3) (1,2) (0,2) (1,1)

[… I]f any of the 4 elementary [canonical] triangles adjacent to the perimeter is removed, the remaining polygon contains a 180-degree angle (*i.e.* a side of length 2).
[… T]his example may be used to generate many others. For instance, substitute the side (3,1)-(2,2) by the sequence

(3,1) (4,2) (5,1) (6,2) (6,3) (5,2) (4,3) (3,2) (2,2)

and attach this [new] PC to its image reflected on the vertical line x = 6. [… T]he result is also "inconstructible" – the only elementary triangles that can be removed without self-intersection are those with vertices at (1,0), (0,2) or (2,3) and their reflections, but they create 180-degree angles.

7. THEOREM ON CONVEX PCs

This proof is due to QUANDT [1].

THEOREM: If a CP is convex it has at most eight sides.
Proof: The sum (of the measures) of the internal angles of a convex polygon with n sides (CP or not) is $(n-2)\pi$. For a CP, the maximum value of this sum is $n(\frac{3\pi}{4})$; therefore:

$$3n\left(\frac{\pi}{4}\right) \geq (n-2)\pi$$
$$3n \geq 4n - 8$$
$$n \leq 8$$

□

COROLLARY: There are only ten possible angular configurations for a convex CP.
Proof: In a convex CP with n sides, let x be the number of internal angles with measure $\pi/4$, y the number of internal angles with measure $\pi/2$ and z the … $3\pi/4$. Then

$$x\left(\frac{\pi}{4}\right) + y\left(\frac{\pi}{2}\right) + z\left(\frac{3\pi}{4}\right) = (n-2)\pi$$

So we have

$$x + y + z = n$$
$$x + 2y + 3z = 4n - 8$$
$$3 \leq n \leq 8$$

(from the previous theorem)
This system has exactly ten solutions, namely:

n	x	y	z
3	2	1	0
4	0	4	0
4	1	2	1
4	2	0	2
5	0	3	2
5	1	1	3
6	1	0	5
6	0	2	4
7	0	1	6
8	0	0	8

Now the analysis of each case […] eliminates the undesirable ones and leads to the eight known CPs.

8. ONLINE MENTIONS OF PCs

Canonical polygons are mentioned in Eric Weisstein's World of Mathematics [3]. The number of PCs of *n* sides is sequence A052436 in the On-Line Encyclopedia of Integer Sequences [4].

References
[1] QUANDT, Waldir (Universidade Federal de Santa Catarina, Florianópolis / SC, Brasil), personal communications 1999-11-16 and 1999-11-23
[2] STOLFI, Jorge (Instituto de Computação, Universidade Estadual de Campinas, Campinas / SP, Brasil), personal communication 2000-06
[3] Eric Weisstein's World of Mathematics – Canonical Polygon – mathworld.wolfram.com/CanonicalPolygon.html
[4] On-Line Encyclopedia of Integer Sequences – A052436 – Canonical polygons of n sides – oeis.org/A052436

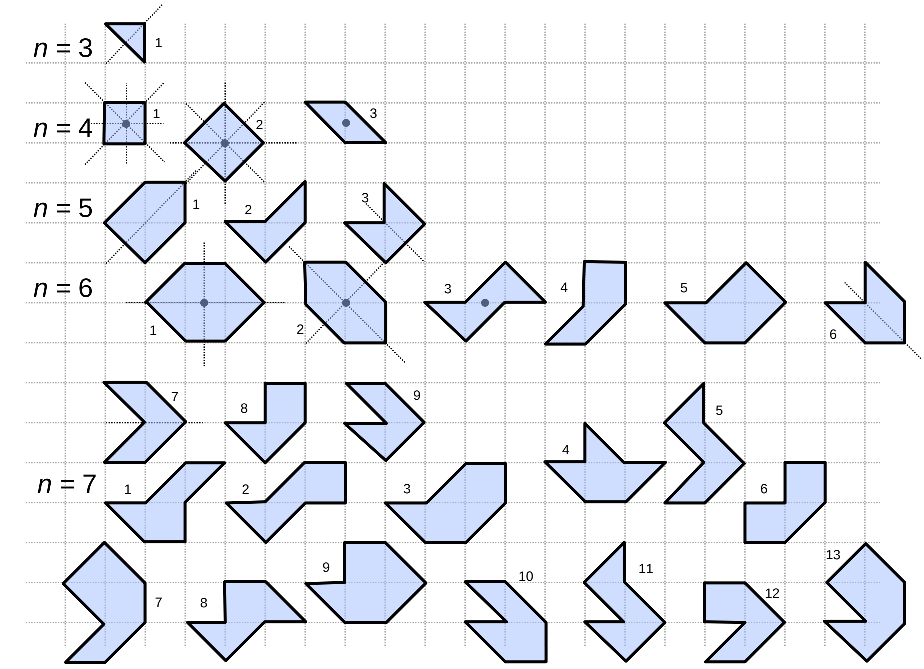
APPENDIX: CANONICAL POLYGON CATALOGUE

The *Canonical Polygon Catalogue* contains:

- A table showing the shape- and measure-dependent properties of the CPs (up to *n* = 8) ordered according to ***Fa***;
- The graphical representation (up to *n* = 8) normalised (direction of rotation, taking angles in the order of the ***Fa***, is clockwise; the first side of the first angle points to the east or north-east), indicating axes and centres of symmetry, and duality.

n	***#***	***Fa***		***Du***	***K***	***Fd***	***Sm***	***Da***	***P***	***D***	***A / D***	***P / D***	***C***			***Q***		
3	1	211	o		0	1110	A	1	3,414	1,414	0,354	2,414	1,000	=	1	0,000	=	0
4	1	2.222	o	+	0	2200	A4	2	4,000	1,414	0,707	2,828	1,000	=	1	1,000	=	1
4	2	2.222	d	-	0	0022	A4	4	5,657	2,000	1,000	2,828	1,000	=	1	0,000	=	0
4	3	3.131	d		0	2020	C	2	4,828	2,236	0,447	2,159	1,000	=	1	0,000	=	0
5	1	32.322	o		0	1121	A	5	6,243	2,236	1,118	2,792	1,000	=	1	0,400	=	2/5
5	2	51.321	d		1	1121		3	6,243	2,236	0,671	2,792	0,750	=	3/4	0,000	=	0
5	3	61.221	o		1	1121	A	3	6,243	2,000	0,750	3,121	0,750	=	3/4	0,000	=	0
6	1	332.332	o	+	0	2022	A2	8	7,657	3,000	1,333	2,552	1,000	=	1	0,500	=	1/2
6	2	332.332	d	-	0	2220	A2	6	6,828	2,828	1,061	2,414	1,000	=	1	0,667	=	2/3
6	3	521.521	d		2	2022	C	4	7,657	3,000	0,667	2,552	0,667	=	2/3	0,000	=	0
6	4	522.331	o	+	1	2220		4	6,828	2,828	0,707	2,414	0,800	=	4/5	0,500	=	1/2
6	5	522.331	d	-	1	2022		6	7,657	3,000	1,000	2,552	0,857	=	6/7	0,333	=	1/3
6	6	613.231	o	+	1	2220	A	4	6,828	2,236	0,894	3,054	0,800	=	4/5	0,500	=	1/2
6	7	613.231	d	-	1	2022	A	4	7,657	2,236	0,894	3,424	0,667	=	2/3	0,000	=	0
6	8	622.321	o		1	2211		4	6,828	2,236	0,894	3,054	0,800	=	4/5	0,500	=	1/2
6	9	713.221	d		1	2031		4	7,657	2,236	0,894	3,424	0,800	=	4/5	0,000	=	0
7	1	5.315.231	d		2	3121		5	8,243	3,162	0,791	2,607	0,714	=	5/7	0,400	=	2/5
7	2	5.322.521	d		2	3121		5	8,243	3,162	0,791	2,607	0,714	=	5/7	0,400	=	2/5
7	3	5.323.331	d		1	3121		7	8,243	3,162	1,107	2,607	0,875	=	7/8	0,571	=	4/7
7	4	6.151.331	o		2	3121		5	8,243	3,000	0,833	2,748	0,714	=	5/7	0,400	=	2/5
7	5	6.215.231	d		2	1132		5	9,071	3,162	0,791	2,869	0,625	=	5/8	0,000	=	0
7	6	6.223.322	o		1	3310	A	5	7,414	2,828	0,884	2,621	0,833	=	5/6	0,800	=	4/5
7	7	6.223.331	d		1	1132		7	9,071	3,162	1,107	2,869	0,778	=	7/9	0,286	=	2/7
7	8	6.231.521	o		2	3121		5	8,243	3,000	0,833	2,748	0,714	=	5/7	0,400	=	2/5
7	9	6.232.331	o		1	3121		7	8,243	3,000	1,167	2,748	0,875	=	7/8	0,571	=	4/7
7	10	7.133.231	d		1	3130		5	8,243	2,828	0,884	2,914	0,833	=	5/6	0,400	=	2/5
7	11	7.215.221	d		2	1132		5	9,071	3,000	0,833	3,024	0,714	=	5/7	0,000	=	0
7	12	7.223.231	o		1	3121		5	8,243	2,236	1,118	3,686	0,833	=	5/6	0,400	=	2/5
7	13	7.223.321	d		1	1132		7	9,071	3,000	1,167	3,024	0,875	=	7/8	0,286	=	2/7

n	*#*	*Fa*	*Du*	*K*	*Fd*	*Sm*	*Da*	*P*	*D*	*A / D*	*P / D*	*C*	*Q*
8	1	33.333.333 od	A	0	2222	A4	14	9,657	3,162	2,214	3,054	1,000 = 1	0,714 = 5/7
8	2	51.515.151 od	A	4	2222	C	6	9,657	3,162	0,949	3,054	0,600 = 3/5	0,333 = 1/3
8	3	52.252.251 od	A	3	2222		8	9,657	3,162	1,265	3,054	0,727 = 8/11	0,500 = 1/2
8	4	52.325.151 od	A	3	2222		8	9,657	3,162	1,265	3,054	0,727 = 8/11	0,500 = 1/2
8	5	52.325.232 od	A	2	2222	C	10	9,657	3,162	1,581	3,054	0,833 = 10/12	0,600 = 3/5
8	6	52.332.522 o	+	2	2222	A	10	9,657	3,162	1,581	3,054	0,833 = 10/12	0,600 = 3/5
8	7	52.332.522 d	-	2	2222	A	10	9,657	3,162	1,581	3,054	0,833 = 10/12	0,600 = 3/5
8	8	52.333.251 od	A	2	2222		10	9,657	3,162	1,581	3,054	0,833 = 10/12	0,600 = 3/5
8	9	52.333.332 od	A	1	2222		12	9,657	3,162	1,897	3,054	0,923 = 12/13	0,667 = 2/3
8	10	53.153.151 d		3	2231		6	9,657	3,606	0,832	2,678	0,667 = 2/3	0,333 = 1/3
8	11	53.153.232 d		2	2231		8	9,657	3,606	1,109	2,678	0,800 = 4/5	0,500 = 1/2
8	12	53.225.322 o	+	2	2042	C	10	10,485	4,123	1,213	2,543	0,833 = 10/12	0,400 = 2/5
8	13	53.225.322 d	-	2	4220	C	6	8,828	3,606	0,832	2,449	0,750 = 3/4	0,667 = 2/3
8	14	53.315.331 o	+	2	2240	C	6	9,657	4,243	0,707	2,276	0,750 = 3/4	0,333 = 1/3
8	15	53.315.331 d	-	2	4022	C	8	9,657	4,000	1,000	2,414	0,800 = 4/5	0,500 = 1/2
8	16	55.132.521 d		2	2222		6	9,657	3,162	0,949	3,054	0,600 = 3/5	0,333 = 1/3
8	17	55.133.331 o	+	1	2222	A	6	9,657	3,162	0,949	3,054	0,600 = 3/5	0,333 = 1/3
8	18	55.133.331 d	-	1	2222	A	8	9,657	3,162	1,265	3,054	0,727 = 8/11	0,500 = 1/2
8	19	61.613.331 od	A	2	2222		6	9,657	3,162	0,949	3,054	0,667 = 2/3	0,333 = 1/3
8	20	62.152.512 d		3	2033	A	6	10,485	4,000	0,750	2,621	0,600 = 3/5	0,000 = 0
8	21	62.233.512 d		2	2033		8	10,485	4,000	1,000	2,621	0,727 = 8/11	0,250 = 1/4
8	22	62.261.331 o		2	4211	A	6	8,828	3,000	1,000	2,943	0,750 = 3/4	0,667 = 2/3
8	23	62.315.322 o		2	4211		6	8,828	3,162	0,949	2,792	0,750 = 3/4	0,667 = 2/3
8	24	62.316.231 o	+	2	4220	C	6	8,828	3,000	1,000	2,943	0,750 = 3/4	0,667 = 2/3
8	25	62.316.231 d	-	2	2042	C	6	10,485	3,606	0,832	2,908	0,600 = 3/5	0,000 = 0
8	26	62.331.621 o		2	2231		6	9,657	3,162	0,949	3,054	0,667 = 2/3	0,333 = 1/3
8	27	63.151.521 o		3	2222		6	9,657	3,162	0,949	3,054	0,667 = 2/3	0,333 = 1/3
8	28	63.152.331 o	+	2	2222		8	9,657	3,162	1,265	3,054	0,800 = 4/5	0,500 = 1/2
8	29	63.152.331 d	-	2	2222		6	9,657	3,000	1,000	3,219	0,667 = 2/3	0,333 = 1/3
8	30	63.216.321 o		2	2222	C	6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	31	63.225.231 o	+	2	2222		8	9,657	3,162	1,265	3,054	0,800 = 4/5	0,500 = 1/2
8	32	63.225.231 d	-	2	2222		6	9,657	3,162	0,949	3,054	0,667 = 2/3	0,333 = 1/3
8	33	63.232.521 o		2	2222		8	9,657	3,162	1,265	3,054	0,800 = 4/5	0,500 = 1/2
8	34	63.233.331 o	+	1	2222		10	9,657	3,162	1,581	3,054	0,909 = 10/11	0,600 = 3/5
8	35	63.233.331 d	-	1	2222		8	9,657	3,162	1,265	3,054	0,800 = 4/5	0,500 = 1/2
8	36	63.315.321 o		2	2231		6	9,657	3,606	0,832	2,678	0,750 = 3/4	0,333 = 1/3
8	37	65.133.321 o		1	2231		6	9,657	3,162	0,949	3,054	0,667 = 2/3	0,333 = 1/3
8	38	66.123.321 o		1	2222	A	6	9,657	3,000	1,000	3,219	0,750 = 3/4	0,333 = 1/3
8	39	71.351.331 d		2	4031		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	40	71.513.331 o		2	2231		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	41	71.613.321 d		2	2231		6	9,657	3,000	1,000	3,219	0,750 = 3/4	0,333 = 1/3
8	42	72.153.231 d		2	2231		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	43	72.162.321 d		2	2222		6	9,657	3,000	1,000	3,219	0,750 = 3/4	0,333 = 1/3
8	44	72.217.221 d		2	2042	C	6	10,485	3,000	1,000	3,495	0,750 = 3/4	0,000 = 0
8	45	72.232.512 o		2	2222		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	46	72.233.322 od	A	1	2222		8	9,657	3,162	1,265	3,054	0,889 = 8/9	0,500 = 1/2
8	47	72.315.231 o		2	2231		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	48	72.316.221 d		2	2033		6	10,485	3,162	0,949	3,316	0,667 = 2/3	0,000 = 0
8	49	72.323.331 o		1	2231		8	9,657	3,162	1,265	3,054	0,889 = 8/9	0,500 = 1/2
8	50	73.152.321 d		2	2231		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	51	73.225.221 d		2	2231		6	9,657	3,162	0,949	3,054	0,750 = 3/4	0,333 = 1/3
8	52	73.233.321 d		1	2231		8	9,657	3,162	1,265	3,054	0,889 = 8/9	0,500 = 1/2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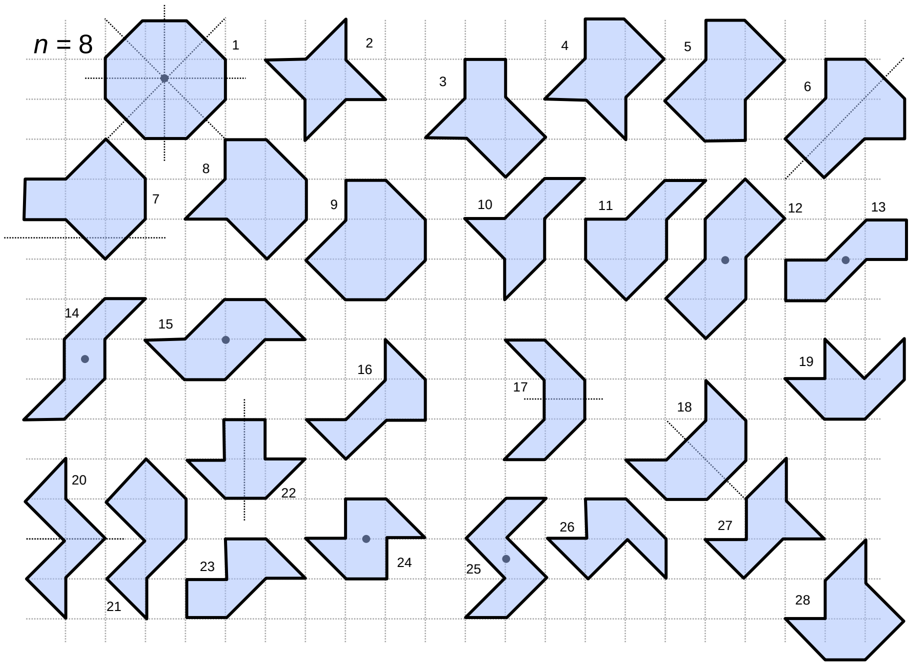
n = 7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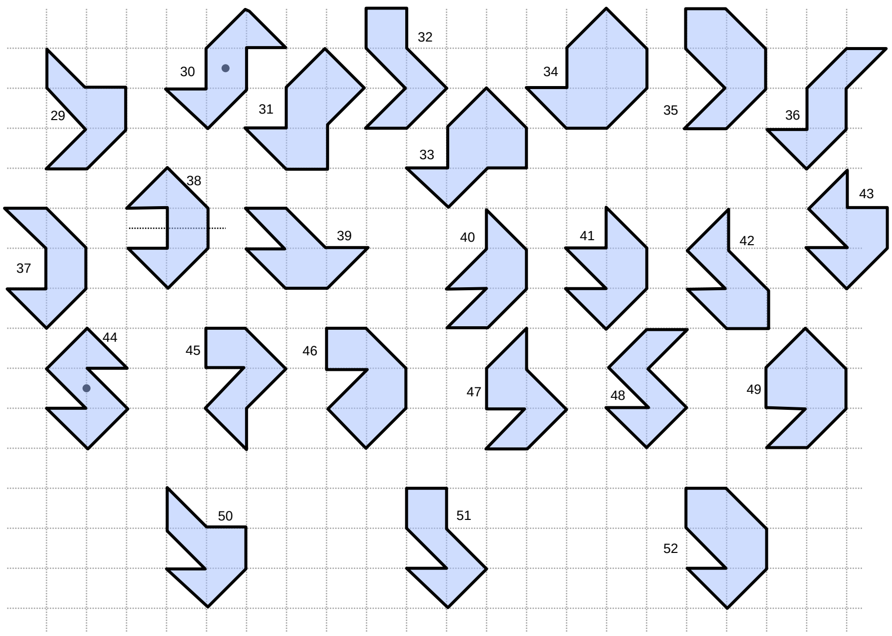

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2021-08