Notes on OEIS A051442 https://oeis.org/A051442

a(n) = $n^{n+1}+(n+1)^n$ for n ≥ 0

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a(n) = A093898(n+1, n), n ≥ 1 https://oeis.org/A093898
a(n) = a(n–1) + A258389(n), n ≥ 1 https://oeis.org/A258389
a(n) = A007778(n) + A000169(n+1) https://oeis.org/A007778
https://oeis.org/A000169

Compare:
https://oeis.org/A076980 Leyland numbers
https://oeis.org/A173054 Numbers of the form a^b+b^a, a > 1, b > a
https://oeis.org/A208506 p^(p+1) + (p+1)^p, where p = prime(n)

Theorems about divisibility of A051442(n)

I. For all n ≠ 1, a(n) mod 8 = 1.

II. Considering the values of n and a(n) mod 6:
for n mod 6 ∈ {0, 3, 5}, a(n) mod 6 = 1;
for n mod 6 = 1, a(n) mod 6 = 3;
for n mod 6 ∈ {2, 4}, a(n) mod 6 = 5.

III. For n ≥ 0, a(n)–1 is a multiple of n^2.

IV. For n odd and n ≥ 3, a(n)–1 is a multiple of (n+1)^2;
for n even and n ≥ 0, a(n)+1 is a multiple of (n+1)^2.

Theorem I proof.
Considering the powers of m mod 8, we observe the following:
if m ≡ 0 then m^k ≡ 0 for all k ≥ 1;
if m ≡ 1 then m^k ≡ 1 for all k ≥ 0;
if m ≡ 2 then m^k ≡ 0 for all k ≥ 3;
if m ≡ 3 then m^k ≡ 1 for all even k and m^k ≡ 3 for all odd k, k ≥ 0;
if m ≡ 4 then m^k ≡ 0 for all k ≥ 2;
if m ≡ 5 then m^k ≡ 1 for all even k and m^k ≡ 5 for all odd k, k ≥ 0;
if m ≡ 6 then m^k ≡ 0 for all k ≥ 3;
if m ≡ 7 then m^k ≡ 1 for all even k and m^k ≡ 7 for all odd k, k ≥ 0.

The cases n=0 and n=2 are trivial: a(0) = 1 and a(2) = 17 which are congruent to 1 mod 8. The theorem does not apply to the case n=1. So now suppose n ≥ 3 and consider a(n) mod 8:
if n ≡ 0 then a(n) = n^(n+1) + (n+1)^n ≡ 0 + 1 ≡ 1;
if n ≡ 1 then a(n) = n^(n+1) + (n+1)^n ≡ 1 + 0 ≡ 1;
if n ≡ 2 then a(n) = n^(n+1) + (n+1)^n ≡ 0 + 1 ≡ 1 (because n is even);

if n ≡ 3 then a(n) = n^(n+1) + (n+1)^n ≡ 1 + 0 ≡ 1 (because n+1 is even);
if n ≡ 4 then a(n) = n^(n+1) + (n+1)^n ≡ 0 + 1 ≡ 1 (because n is even);
if n ≡ 5 then a(n) = n^(n+1) + (n+1)^n ≡ 1 + 0 ≡ 1 (because n+1 is even);
if n ≡ 6 then a(n) = n^(n+1) + (n+1)^n ≡ 0 + 1 ≡ 1 (because n is even);
if n ≡ 7 then a(n) = n^(n+1) + (n+1)^n ≡ 1 + 0 ≡ 1 (because n+1 is even).

Therefore a(n) mod 8 = 1 for all n ≠ 1. Q.E.D.

Theorem II proof.
Considering the powers of m mod 6, we observe the following:
if m ≡ 0 then m^k ≡ 0 for all k ≥ 1;
if m ≡ 1 then m^k ≡ 1;
if m ≡ 2 then m^k ≡ 4 for k even and k ≥ 2, m^k ≡ 2 for k odd;
if m ≡ 3 then m^k ≡ 3 for all k ≥ 1;
if m ≡ 4 then m^k ≡ 4 for all k ≥ 1;
if m ≡ 5 then m^k ≡ 1 for k even, m^k ≡ 5 for k odd.

For the cases n=0 and n=1, we have a(0) = 1 and a(1) = 3, which satisfy the proposition to be proved. Now suppose n > 1 and consider n and a(n) mod 6:
if n ≡ 0 then a(n) = n^(n+1) + (n+1)^n ≡ 0 + 1 ≡ 1;
if n ≡ 1 then a(n) = n^(n+1) + (n+1)^n ≡ 1 + 2 ≡ 3 (because n is odd);
if n ≡ 2 then a(n) = n^(n+1) + (n+1)^n ≡ 2 + 3 ≡ 5 (because n+1 is odd);
if n ≡ 3 then a(n) = n^(n+1) + (n+1)^n ≡ 3 + 4 ≡ 1;
if n ≡ 4 then a(n) = n^(n+1) + (n+1)^n ≡ 4 + 1 ≡ 5 (because n is even);
if n ≡ 5 then a(n) = n^(n+1) + (n+1)^n ≡ 1 + 0 ≡ 1 (because n+1 is even).

Therefore, considering the values of n and a(n) mod 6:
for n ≡ 0, 3, or 5, a(n) ≡ 1;
for n ≡ 1, a(n) ≡ 3;
for n ≡ 2 or 4, a(n) ≡ 5.
Q.E.D.

Theorem III proof.
For n = 0, 1, or 2 we have:
a(0) – 1 = 0, which is a multiple of 0^2;
a(1) – 1 = 2, which is a multiple of 1^2;
a(2) – 1 = 16. which is a multiple of 2^2.
Now suppose n > 2 and consider the binomial expansion of (n+1)^n:

$$n^n+\binom{n}{1}n^{n-1}+\binom{n}{2}n^{n-2}+...+\binom{n}{n-2}n^2+\binom{n}{n-1}n+1$$

The penultimate term, $\binom{n}{n-1}n$, is equal to n^2. Every term to the left of that one is a multiple of n^2. It’s only the rightmost term, 1, that is not a multiple of n^2. Therefore we have (n+1)^n ≡ 1 mod n^2.

Because n > 2, we can say n^(n+1) ≡ 0 mod n^2.

Now a(n) – 1 = n^(n+1) + (n+1)^n – 1 ≡ 0 + 1 – 1 ≡ 0 mod n^2.

Therefore for all n ≥ 0, a(n)–1 is a multiple of n^2. Q.E.D.

Theorem IV proof.
For n=0 we have a(0) + 1 = 2, which is a multiple of 1^2. The case n=1 is excluded from this theorem. So now suppose n ≥ 2. Let m = n+1.

Consider $(m-1)^m \, mod \, m^2$. First look at the binomial expansion of (m – 1)^m:

$$m^m-\binom{m}{1}m^{m-1}+\binom{m}{2}m^{m-2}-...\pm\binom{m}{m-2}m^2\pm\binom{m}{m-1}m\pm 1$$

The rightmost term in this expansion is +1 if m is even, and –1 if m is odd. The penultimate term, $\pm\binom{m}{m-1}m$, is ±m^2. All the terms to the left of that one are multiples of m^2. So we have (m–1)^m ≡ 1 if m is even, –1 if m is odd, mod m^2.

Also, m^(m–1) ≡ 0 mod m^2. (We can say this because m ≥ 3, since n ≥ 2 and m=n+1.)

Therefore (m–1)^m + m^(m–1) ≡ +1 if m is even, –1 if m is odd, mod m^2.

And since m=n+1, we now have:

a(n) ≡ +1 if n is odd, –1 if n is even, mod (n+1)^2, for all n > 2.

Therefore:
  For n odd and n > 2, a(n)–1 is a multiple of (n+1)^2;
  for n even and n ≥ 0, a(n)+1 is a multiple of (n+1)^2.
Q.E.D.